Positivity of linear maps under tensor powers [arXiv:1502.05630]

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Existence for each $n \in \mathbb{N}$

For any dimensions $d_1, d_2 \geq 2$ and any $n \in \mathbb{N}$, there exists a *non-trivial* n-tensor-stable positive map $\mathcal{P} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$.

Constructive proof

$\mathrm{Choi}_{\mathcal{P}} := C \in \mathcal{M}_{d_1} \otimes \mathcal{M}_{d_2}$ should satisfy

$$\left(\langle\Psi| \otimes \langle\Phi|\right) C^{\otimes n} \left(|\Psi\rangle \otimes |\Phi\rangle\right) \geq 0 \qquad (1)$$

for all $|\Psi\rangle$ and $|\Phi\rangle$, but C should neither be positive semidefinite nor PPT.

Idea: Let $C := P - \varepsilon\mathbb{I}$, where $P \geq 0$ and

(a) $\mathrm{rank}[P]$, $\mathrm{rank}[P^\Gamma] < d_1 d_2$,

(b) $0 < \inf_{\psi,\phi} \langle\psi|\langle\phi|P|\psi\rangle|\phi\rangle =: \mu$,

(c) $P = \sum_i A_i \otimes B_i$ is separable ($A_i, B_i \geq 0$).

⇝ can *explicitly write down* such P from some (non-orthogonal) unextendable product set.

Property (c) ensures **multiplicativity**:

$$\inf_{\Psi,\Phi} \langle\Psi|\langle\Phi|P^{\otimes n}|\Psi\rangle|\Phi\rangle = \mu^n > 0,$$

and Eq. (1) follows for $C^{\otimes n} = (P - \varepsilon\mathbb{I})^{\otimes n}$. □

Relate to entanglement distillation

If $\mathcal{P}$ is $(n+1)$-tensor-stable positive, then

$$\frac{d_{\mathrm{CP}}(\mathcal{P})}{\|\mathcal{P}\|_\diamond} \leq \inf_{\mathcal{S}\in\mathrm{LOCC}} \left\|\Omega_{d_1} - \mathcal{S}(\mathrm{Choi}_{\mathcal{P}}^{\otimes n})\right\|_1,$$

where $d_{\mathrm{CP}}(\mathcal{P}) := \frac{1}{2}\|\mathrm{Choi}_{\mathcal{P}}\|_1 - \frac{1}{2}\mathrm{tr}[\mathrm{Choi}_{\mathcal{P}}]$.

Idea: If $\Omega_{d_1} \approx \mathcal{S}(\mathrm{Choi}_{\mathcal{P}}^{\otimes n})$, then $\exists \mathcal{E}_i, \mathcal{D}_i \in \mathrm{CP}$:

$$\mathrm{id}_{d_1} \approx \sum_i \mathcal{D}_i \circ \mathcal{P}^{\otimes n} \circ \mathcal{E}_i.$$

Therefore, since $\mathcal{P}^{\otimes(n+1)} \geq 0$:

$$\mathrm{id}_{d_1} \otimes \mathcal{P} \approx \Big(\sum_i \mathcal{D}_i \circ \mathcal{P}^{\otimes n} \circ \mathcal{E}_i\Big) \otimes \mathcal{P} = \sum_i (\mathcal{D}_i \otimes \mathrm{id}) \circ \mathcal{P}^{\otimes(n+1)} \circ (\mathcal{E}_i \otimes \mathrm{id}) \geq 0.$$

If approximation good, then $\mathcal{P}$ is "close" to CP. □

Implication: If $\mathcal{P} \notin \mathrm{CP}$ is tensor-stable positive, then $\mathrm{Choi}_{\mathcal{P}}$ is *"undistillable"*.

⇝ apply distillation to **block-positive** operators:

Twirling block-positive matrices

If $\mathcal{P}' : \mathcal{M}_d \to \mathcal{M}_d$ positive with $\mathrm{tr}[\mathbb{F}\,\mathrm{Choi}_{\mathcal{P}'}] \leq 0$,

then: $\int_{U\in\mathcal{U}(d)} (U \otimes U)\,\mathrm{Choi}_{\mathcal{P}'}\,(U \otimes U)^\dagger\, dU \geq 0.$

Definition: Tensor-stable positive map

- $\mathcal{P} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$ is **n-tensor-stable positive** (for some $n \in \mathbb{N}$) $:\Leftrightarrow$ $\mathcal{P}^{\otimes n} : \mathcal{M}_{d_1^n} \to \mathcal{M}_{d_2^n}$ is positive.
- $\mathcal{P} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$ is **tensor-stable positive** $:\Leftrightarrow$ $\mathcal{P}$ is n-tensor-stable positve $\forall n \in \mathbb{N}$.

Trivial examples:
- Any **completely positive map** $\mathcal{P} = \mathcal{T} \in \mathrm{CP}$.
- Any **completely co-positive map** $\mathcal{P} = \Theta \circ \mathcal{T} \in \mathrm{coCP}$ (Θ = **transposition**)

General existence results

Tensor-stable positive ⇒ NPT-BE

If $\mathcal{P} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$ is a *non-trivial* tensor-stable positive map, then $\exists$ NPT bound entanglement in dimensions $d_1 \otimes d_1$ and in $d_2 \otimes d_2$.

Proof: (1) Filtering: $(\mathrm{Choi}_{\mathcal{P}})^\Gamma \not\geq 0 \Rightarrow \exists A$ s.th.

$$C := (A^\dagger \otimes \mathbb{I})\,\mathrm{Choi}_{\mathcal{P}}\,(A \otimes \mathbb{I})$$

is block-positive and satisfies $\mathrm{tr}[\mathbb{F}\,C] < 0$.

(2) Twirling: Thus, the LOCC action

$$\mathcal{S}(\mathrm{Choi}_{\mathcal{P}}) := \frac{1}{\mathrm{tr}[C]} \int_U (U \otimes U)\, C\, (U \otimes U)^\dagger\, dU$$

yields an NPT Werner state. We show next that it is *not* distillable, i.e. it is bound entangled.

(3) Contradiction: If it were distillable, then $\|\Omega_{d_1} - \mathcal{S}_n \circ \mathcal{S}^{\otimes n}(\mathrm{Choi}_{\mathcal{P}}^{\otimes n})\|_1 \to 0$ as $n \to \infty$. But according to above statement, then $\mathcal{P} \in \mathrm{CP}$ ↯ □

Corollary: qubit in- or outputs

If $\mathcal{P} : \mathcal{M}_2 \to \mathcal{M}_{d_2}$ or $\mathcal{P} : \mathcal{M}_{d_1} \to \mathcal{M}_2$ is tensor-stable positive, then $\mathcal{P} \in (\mathrm{CP} \cup \mathrm{coCP})$.

(since every qubit NPT Werner state is distillable)

One-parameter candidate families

A non-trivial tensor-stable positive map $\mathcal{P}$ exists **iff** one exists within the family $\mathcal{W}_p \otimes (\Theta \circ \mathcal{W}_p)$, where $\mathrm{Choi}_{\mathcal{W}_p} \in \{\text{NPT Werner states}\}_{p\in[-1,0)}$:

$$\mathcal{W}_p(\rho) := (d-p)\mathrm{tr}[\rho]\,\mathbb{I}_d - (1-dp)\rho^T.$$

Proof: Filter (1), twirl (2), and depolarize, such that the LOCC actions $\mathcal{S}_1(\mathrm{Choi}_{\mathcal{P}})$ and $\mathcal{S}_2(\mathrm{Choi}_{\mathcal{P}})$ are the Choi matrices of $\mathcal{W}_p$ resp. $\Theta \circ \mathcal{W}_p$.

Therefore, when $\mathcal{P}$ is n-tensor-stable positive, then $\mathcal{W}_p \otimes (\Theta \circ \mathcal{W}_p)$ is $\lfloor n/2 \rfloor$-tensor-stable positive. □

Quantum capacity bounds

Let $\mathcal{P} : \mathcal{M}_{d_3} \to \mathcal{M}_{d_2}$ be tensor-stable positive, surjective, unital, but *not* CP.

Upper bound on $\mathcal{Q}(\mathcal{T})$

For any quantum channel $\mathcal{T} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$:

$$\mathcal{Q}(\mathcal{T}) \leq \frac{\log\left(\|\mathcal{P}^{-1} \circ \mathcal{T}\|_\diamond \|\mathcal{P}^*(\mathbb{I})\|_\infty\right) \log d_2}{\log \|\mathcal{P}^*\|_\diamond}$$

Proof: Generalize Holevo & Werner (2001): If

$$\frac{1}{2}\|\mathrm{id}_{d_2}^{\otimes n} - \mathcal{D} \circ \mathcal{T}^{\otimes m} \circ \mathcal{E}\|_\diamond \leq \varepsilon,$$

then by multiplying $(\Theta \circ \mathcal{P}^* \circ \Theta)^{\otimes n}$ from left:

$$(1-2\varepsilon)\,\|\Theta \circ \mathcal{P}^* \circ \Theta\|_\diamond^n \leq \|\mathcal{P}^*(\mathbb{I})\|_\infty^m \|\mathcal{P}^{-1} \circ \mathcal{T}\|_\diamond^m.$$

New insight needed here:

$$\left((\Theta \circ \mathcal{P}^* \circ \Theta)^{\otimes n} \circ \mathcal{D} \circ \mathcal{P}^{\otimes m}\right) \otimes \mathrm{id}\,(\Omega) = (\Theta \circ \mathcal{P}^* \circ \Theta)^{\otimes(n+m)} \circ (\mathcal{D} \otimes \mathrm{id})\,(\Omega) \geq 0.$$

⇝ completely positive ⇝ cb-norm $= \|\mathcal{P}^*(\mathbb{I})\|_\infty^m$.

Quantum capacity: $\mathcal{Q}(\mathcal{T}) = \sup \frac{n \log d_2}{m}$. □

⇝ **"pretty strong converse"-rate:** any $\varepsilon < \frac{1}{2}$!

Special case $\mathcal{P} = \Theta$: **cb-norm bound**

$$\mathcal{Q}(\mathcal{T}) \leq \log \|\Theta \circ \mathcal{T}\|_\diamond.$$

- better than above whenever $\mathcal{P} \in \mathrm{coCP}$, but:
- $\mathcal{P} \notin \mathrm{coCP}$ yields *new* channels with $\mathcal{Q}_2(\mathcal{T}) = 0$:

LOCC-assisted $\mathcal{Q}_2(\mathcal{T}) = 0$

Let $\mathcal{P}$ be tensor-stable positive, but *not* CP.

Let $\mathcal{T} = \sum_j \mathcal{R}_j \circ \mathcal{P} \circ \mathcal{S}_j$ with $\mathcal{R}_j, \mathcal{S}_j \in \mathrm{CP}$.

Then: $\mathcal{Q}_2(\mathcal{T}) = \mathcal{Q}(\mathcal{T}) = 0$.

Proof: Similar as above:

$$\frac{d_{\mathrm{CP}}(\mathcal{P})}{\|\mathcal{P}\|_\diamond} \leq \inf_{\mathcal{E}_i,\mathcal{D}_i \in \mathrm{CP}} \Big\|\mathrm{id}_{d_3} - \sum_i \mathcal{D}_i \circ \mathcal{T}^{\otimes m} \circ \mathcal{E}_i\Big\|_\diamond. \quad \square$$

- "pretty strong converse"-error $\leq \frac{1}{2} d_{\mathrm{CP}}(\mathcal{P})/\|\mathcal{P}\|_\diamond$
- if $\mathcal{P} \notin (\mathrm{CP} \cup \mathrm{coCP})$, then $\mathcal{Q}_2(\mathcal{W}_p) = 0$ for **some NPT channel** $\mathcal{W}_p$ (already anti-degradable)

Strong converse rates

Stronger result for $\mathcal{P} = \Theta$ (involution!):

Strong converse rate for $\mathcal{Q}_2$

Error of LOCC-scheme for id_N using m times $\mathcal{T}$:

$$\varepsilon \geq 1 - \frac{\|\Theta \circ \mathcal{T}\|_\diamond^m}{N}.$$

Thus: (i) $\mathcal{Q}_2(\mathcal{T}) \leq \log \|\Theta \circ \mathcal{T}\|_\diamond$.

(ii) is (exponential) strong converse rate.

Proof: Fidelity $f := \mathrm{tr}\left[\Omega_N\,(\mathcal{L}_{\mathrm{LOCC}} \otimes \mathrm{id}_N)(\Omega_N)\right]$.

(a) $\|\Theta \circ \mathcal{T}\|_\diamond^m \geq \|\Theta \circ \mathcal{L}_{\mathrm{LOCC}(m\times\mathcal{T})}\|_\diamond \geq \left\|\left((\Theta \circ \mathcal{L}) \otimes \mathrm{id}_N\right)(\Omega_N)\right\|_1 \overset{\text{twirl}}{\geq} Nf.$

(b) $\varepsilon := \frac{1}{2}\|\mathrm{id}_N - \mathcal{L}_{\mathrm{LOCC}(m\times\mathcal{T})}\|_\diamond \geq \frac{1}{2}\|\Omega_N - (\mathcal{L} \otimes \mathrm{id}_N)(\Omega_N)\|_1 \overset{\text{twirl}}{\geq} 1 - f.$ □

For any tensor-stable positive, surjective $\mathcal{P} \notin \mathrm{CP}$:

Strong converse rate for $\mathcal{Q}$

Let $\mathcal{P} : \mathcal{M}_{d_3} \to \mathcal{M}_{d_2}$ be tensor-stable positive, unital, surjective, but *not* CP. **Then:**

$$\frac{\log\left(\|\mathcal{P}^{-1} \circ \mathcal{T}\|_\diamond \|\mathcal{P}^*(\mathbb{I})\|_\infty\right) \log d_2}{\log \|(\mathcal{P}^* \otimes \mathrm{id}_{d_2})(\Omega_{d_2})\|_1}$$

is (exponential) strong converse rate for $\mathcal{Q}(\mathcal{T})$.

Proof: Again, track fidelity of the maximally entangled state $\Omega_{d_2}^{\otimes n}$. □

⇝ worse than upper bound on $\mathcal{Q}(\mathcal{T})$ above

Questions

- **Do non-trivial ts-positive maps exist?**
  ⇝ one-parameter candidate families
  ⇝ can these prove qubit-Corollary directly?
- Converse: NPT-BE ⇒ tensor-stable positive?
  ⇝ compare to NPT-BE via 2-positive maps: the coCP maps are *trivial* in our case
- Note: unital $\mathcal{P}$ is n-tensor-stable positive $\Leftrightarrow$ $\|\mathcal{P}^{\otimes n}\|_{\infty\to\infty} = 1$.
- True? $\|\Theta \circ (\mathrm{id} - \mathcal{T})\|_\diamond \leq \frac{1}{2}\|\Theta\|_\diamond \|\mathrm{id} - \mathcal{T}\|_\diamond$?
  ⇝ would get strong converse via Holevo & Werner
  Note: $\|\Theta_d\|_\diamond = \|(\Theta_d \otimes \mathrm{id}_d)(\Omega_d)\|_1 = d$.
- New channels with $\mathcal{Q}(\mathcal{T}) = 0$ from $\mathcal{P}$?
  $\mathcal{Q}_2$-strong converse rates from $\mathcal{P}$?