Motivation

Before I give the outline of these lectures I would like to explain in non-technical terms what is non-relativistic QED. In short, it is a theory at the interface between non-relativistic Quantum Mechanics and relativistic QED.

0.1 Quantum Mechanics

Consider a Hydrogen atom described by \( H = -\Delta_x - \alpha/|x| \). What quantum mechanics teaches about the spectrum of this operator is that it consists of a ground state and excited states

\[
E_n = -\frac{1}{4} \frac{\alpha^2}{n^2}, \quad n = 1, 2, 3...
\] (0.1)

and then a continuous spectrum above zero. This means, that the electron in an excited state would stay in such an excited state forever. But experiments tell you that this is simply not true, the lines have a finite width (even at zero temperature) and thus after some time the electron relaxes to a ground state emitting photons. In the Schrödinger equation you know from Quantum Mechanics there are no terms responsible for this effect. The physical reason is coupling of the electron to the quantized electromagnetic field, which is usually not covered by introductory Quantum Mechanics courses. Relaxation of excited atoms to the ground state is one example of a question which cannot be answered within non-relativistic Quantum Mechanics. Non-relativistic QED offers an appropriate framework to study this question. (Relativistic QED is difficult to apply to bound state problems due to its perturbative character - see below).

0.2 Relativistic Quantum Electrodynamics

Full Quantum Electrodynamics (QED) describes interactions between electrons and positrons, described by the electric current density \( j \), and photons, described by the electromagnetic potential \( A \). The interaction, formally given by

\[
V_{\text{RQED}} = e \int d^3x j(x) A(x),
\] (0.2)
turns out to be very singular, since $j$ and $A$ are distributions, but we ignore these ultraviolet problems here for a moment. Given the interaction, the next step is to write down the scattering matrix. Rules of the game of Quantum Mechanics give

$$ S = \text{Texp} \left( -i \int_{-\infty}^{\infty} dt V_{\text{RQED}}^I(t) \right) \quad (0.3) $$

where $V_{\text{RQED}}^I$ is the interaction in the interaction picture. To compute the probability that a system evolves from some initial state $|\alpha\rangle$ to a final state $|\beta\rangle$ one needs to compute the scattering matrix element $\langle\alpha|S|\beta\rangle$. Let us consider Compton scattering, i.e. collision of one electron and one photon. The probability of a transition $\alpha \rightarrow \beta$ (collision cross section $\sigma$) satisfies

$$ \sigma \sim |\langle\alpha|S|\beta\rangle|^2. \quad (0.4) $$

$\langle\alpha|S|\beta\rangle$ can be computed as a power series in the coupling constant $e$ and the resulting expressions can be depicted as Feynman diagrams which capture the intuitive meaning of the respective contributions. The leading contribution is given by a tree diagram (TREE DIAGRAM). Further contributions involve emission and reabsorption of virtual photons (RADIATIVE CORRECTION DIAGRAM). These contributions have the so called infrared divergences that is divergences at small values of the photon energy. These divergences can be traced back to vanishing mass of the photon and integration over whole space in (0.2). They have to be regularized by introducing an infrared cut-off $\lambda > 0$ (simply eliminating photons of energy smaller than $\lambda$). The resulting $S$-matrix element $\langle\alpha|S^\lambda|\beta\rangle$ can be computed, but $\lim_{\lambda \rightarrow 0} \langle\alpha|S^\lambda|\beta\rangle = 0$ as if there was no scattering. Thus standard rules of Quantum Mechanics give an experimentally unacceptable result: $\sigma = 0$! This is one manifestation of the infrared problem.

A way out, proposed by Yennie, Frautschi and Suura \[3\] is a serious deviation from these rules of the game. We should not consider the process $\alpha \rightarrow \beta$ alone, but a whole family of processes $\alpha \rightarrow \beta_n$, where $\beta_n$ involves emission of $n$ photons of total energy $E_n$ in addition to particles present in $\beta$. (SOFT PHOTON EMISSION DIAGRAM). The resulting inclusive cross-section is given by

$$ \sigma_{\text{inc}}(E_t) \sim \lim_{\lambda \rightarrow 0} \sum_{n=0}^{\infty} |\langle\alpha|S^\lambda|\beta\rangle|^2, \quad (0.5) $$

which is finite and not identically zero. It gives results consistent with experiments if $E_t$ is chosen below the sensitivity of the detector. A mathematically rigorous derivation of this formula from first principles has not been achieved in the perturbative framework of relativistic QED, in spite of many attempts \[2\]. In contrast, in the framework of non-relativistic QED, there has been steady progress in understanding of the infrared problem. Important advantage: availability of a Hamiltonian as a self-adjoint operator on a Hilbert space (and not just a formal power series).

\[1\] $\sigma = \frac{1}{n} \frac{d\Phi}{dz}$, where $d\Phi$ is a loss of the flux due to the event, $dz$ is the thickness of the target material and $n$ is the number density.
The problem of relaxation of Hydrogen atom to the ground state is also difficult to study in the perturbative setting of relativistic QED, because electron confined in an atom cannot be considered a small perturbation of a freely moving electron.

0.3 Non-relativistic Quantum Electrodynamics

Start from the relativistic QED interaction:

\[ V_{\text{RQED}} = e \int d^3x \, j(x)A(x) \]  

(0.6)

1. \( j \) and \( A \) are distributions, problems with pointlike multiplication. We need to regularize: Replace \( j \) with a convolution \( j \ast \varphi \) for a nice function \( \varphi \) so that \( j \ast \varphi \) is now also a nice function. \( \varphi \) plays a role of charge distribution of the electron.

2. Integral over whole \( \mathbb{R}^3 \) difficult to control. It helps to remove terms from \( j \) which are responsible for electron-positron pair creation. The result can be denoted as no-pairs current \( j_{\text{np}} \).

Thus we are left with the interaction which is a controllable expression

\[ V_{\text{NRQED}} = \int d^3x \, (j_{\text{np}} \ast \varphi)(x)A(x). \]  

(0.7)

After some further steps and simplifications one obtains the standard Hamiltonian of non-relativistic QED (Pauli-Fierz Hamiltonian):

\[ H_{\text{NRQED}} = \sum_{j=1}^{N} \frac{1}{2m} \left( p_j - e(A \ast \varphi)(q_j) \right)^2 + V_c(q) + H_f. \]  

(0.8)

where \( (p_j, q_j) \) are positions and momenta of electrons, \( V_c \) is the Coulomb interaction between the electrons and \( H_f \) is the energy of photons. It is a reasonable approximation to full QED in the low energy regime, in particular below the electron-positron pair production threshold. In the next section we will obtain this Hamiltonian along a different route by quantization of classical Maxwell equations coupled to particles.

0.4 Outline of the course

1. Quantization of charged particles interacting with the electromagnetic field.

2. Pauli-Fierz Hamiltonian and other models of non-relativistic QED.

3. Fock spaces and self-adjointness of the Pauli-Fierz Hamiltonian.

1 Quantization of particles interacting with the electromagnetic field

1.1 Equations of motion

Let $E, B, \rho, j$ be the electric field, magnetic field, charge density and current density, respectively. These are functions on space-time satisfying the Maxwell equations (here we set velocity of light $c = 1$):

\[ \partial_t B = -\nabla \times E, \quad (1.9) \]
\[ \partial_t E = \nabla \times B - j, \quad (1.10) \]
\[ \nabla \cdot E = \rho, \quad (1.11) \]
\[ \nabla \cdot B = 0. \quad (1.12) \]

We want $\rho$ and $j$ to describe a collection of $N$ particles of finite extension. Thus we introduce a nice function $\varphi \in S(\mathbb{R}^3)$, s.t. $\varphi(x) = \varphi(-x)$, which models the charge distribution of each particle. Let $t \to q_j(t)$ be the trajectories. The charge and current are given by:

\[ \rho(t, x) = e \sum_{j=1}^{N} \varphi(x - q_j(t)), \quad (1.13) \]
\[ j(t, x) = e \sum_{j=1}^{N} \varphi(x - q_j(t)) \dot{q}_j(t). \quad (1.14) \]

They obviously satisfy the charge conservation equation:

\[ \partial_t \rho(t, x) + \nabla_x j(t, x) = 0. \quad (1.15) \]

The total charge of each particle is

\[ Q := e \int d^3x \varphi(x) = (2\pi)^{3/2}e\hat{\varphi}(0). \quad (1.16) \]

If $Q \neq 0$, the particle will be called an electron. If $Q = 0$ the particle will be called an atom.

We couple this system to the Newton equations of motion

\[ m \ddot{q}_j(t) = e(E_\varphi(t, q_j(t)) + \dot{q}_j(t) \times B_\varphi(t, q_j(t))), \quad (1.17) \]
where
\[ E_\varphi(t, x) = (E \ast \varphi)(t, x) = \int d^3y E(t, x - y)\varphi(y), \]
and similarly for \( B \). Clearly, for \( \varphi \rightarrow \delta \) we have \( E_\varphi(t, x) \rightarrow E(t, x) \) and \( Q = e \) but in this limit the system of equations becomes singular. Thus in general the parameter \( e \) should be interpreted as a coupling constant, which determines the strength of interaction between the fields and the particles, rather than charge.

1.2 Electromagnetic potentials

We introduce the electromagnetic potentials \( \phi, A \), which satisfy
\[ E = -\partial_t A - \nabla \phi, \]
\[ B = \nabla \times A. \]

Since \( \nabla(\nabla \times A) \equiv 0 \) and \( (\nabla \times \nabla \phi) \equiv 0 \), this guarantees
\[ \nabla \cdot B = 0 \quad \text{and} \quad \partial_t B = -\nabla \times E. \]

Note that the potentials \((\phi, A)\) are not unique. For example, for any smooth \( f \), the new potentials \( \tilde{A}(t, x) = A(t, x) + \nabla f(x) \) and \( \tilde{\phi}(t, x) = \phi(t, x) \) give rise to the same fields \( E, B \). (Because \( \nabla \times \nabla f = 0 \)). This is called a change of gauge of \((\phi, A)\). Exploiting gauge freedom, we can impose additional conditions on the potentials. For example, by choosing \( f \) s.t. \( \Delta f = -\nabla \cdot A \) we obtain
\[ \nabla \cdot \tilde{A} = 0. \]
In this case we say that \( \tilde{A} \) satisfies the Coulomb gauge condition.

1.3 Lagrangian formulation

The Lagrangian is given by
\[ L = \frac{1}{2} \sum_{j=1}^{N} m\dot{q}_j^2 + \int d^3x \left( \frac{1}{2}(E(t, x)^2 - B(t, x)^2) + j(t, x) \cdot A(t, x) - \rho(t, x)\phi(t, x) \right) \]
\[ = \frac{1}{2} \sum_{j=1}^{N} m\dot{q}_j^2 + \int d^3x \left( \frac{1}{2}(-\partial_t A - \nabla \phi)^2 - \frac{1}{2}(\nabla \times A)^2 \right. \]
\[ + e \sum_{j=1}^{N} \varphi(x - q_j(t))\dot{q}_j \cdot A - e \sum_{j=1}^{N} \varphi(x - q_j(t))\phi \right), \]
\[ \text{(1.23)} \]
where \( \{ q_j, \phi(x), A(x) \} \) are understood as coordinates. The Euler-Lagrange equations give the remaining equations of motion:

\[
\frac{\partial t}{\partial \dot{q}_j^i} - \frac{\partial L}{\partial q_j^i} = 0 \quad \text{gives} \quad m \ddot{q}_j^i = e \left( E^\phi(t, q(t)) + \dot{\phi} \times B^\phi(t, q(t)) \right), \quad (1.24)
\]

\[
\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0 \quad \text{gives} \quad \nabla E = \rho, \quad (1.25)
\]

\[
\frac{\partial L}{\partial \dot{A}_i^j} - \frac{\partial L}{\partial A_i^j} = 0 \quad \text{gives} \quad \partial_t E = \nabla \times B - j. \quad (1.26)
\]

**Remark 1.1.** In relativistic field theory one often considers action of the form

\[
S = \int d^4 x \, \mathcal{L}(\phi(x), \partial_\mu \phi(x)), \quad (1.27)
\]

where \( x \) is a four-vector. Then the Euler-Lagrangian equations have the form

\[
\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi}. \quad (1.28)
\]

Here \( \mathcal{L} \) is a Lagrangian density (as opposed to the Lagrangian \( L \)) and therefore all the derivatives can be considered partial derivatives (as opposed to the functional derivatives \( \delta / \delta \phi \) computed according to (1.34)).

Let us show (1.24): We have

\[
\frac{\partial L}{\partial \dot{q}_j^i} = m \ddot{q}_j^i + \int d^3 x \, e \phi(x - q_j(t)) A_i^j(t, x), \quad \text{(1.29)}
\]

\[
\frac{\partial L}{\partial q_j^i} = m \dot{q}_j^i - \int d^3 x \, e \partial_k \phi(x - q_j(t)) \dot{q}_j^k A_i^j(t, x)
\]

\[
\quad + \int d^3 x \, e \phi(x - q_j(t)) \partial_i A_i^j(t, x), \quad \text{(1.30)}
\]

\[
\frac{\partial L}{\partial q_j^i} = - \int d^3 x \, e \left( \partial_i \phi(x - q_j(t)) \dot{q}_j^k A^k - \partial_i \phi(x - q_j(t)) \phi \right)
\]

\[
\quad = \int d^3 x \, e \left( \phi(x - q_j(t)) \dot{q}_j^k \partial_i A_i^k - \phi(x - q_j(t)) \partial_i \phi \right) \quad (1.32)
\]

(Summation over index \( k \) understood). To conclude, we use

\[
[\dot{q}_j \times (\nabla \times A)]^i = [\nabla(\dot{q}_j A)]^i - (\dot{q}_j \cdot \nabla) A_i^j = \dot{q}_j^k \partial_i A^k - \dot{q}_j^k \partial_k A_i^j \quad (1.33)
\]

and the property \( \phi(x) = \phi(-x) \).
Let us now show (1.25): The functional derivative \( \frac{\delta}{\delta \phi(t, x)} \) is the 'Frechet derivative in the direction of Dirac \( \delta \)'. The rule of the game is:

\[
\frac{\delta \phi(t, x)}{\delta \phi(t, y)} = \delta(x - y).
\]

(1.34)

Thus we have

\[
\frac{\delta L}{\delta \dot{\phi}(t, y)} = 0
\]

\[
\frac{\delta L}{\delta \phi(t, y)} = \int d^3x \left( -(-\partial_t A(t, x) - \nabla \phi(t, x)) \nabla \delta(x - y) - e \sum_{j=1}^{N} \varphi(x - q_j(t)) \delta(x - y) \right)
\]

\[
= \int d^3x \left( \nabla E(t, x) \delta(x - y) - e \rho(t, x) \delta(x - y) \right)
\]

\[
= \nabla E(t, y) - e \sum_{j=1}^{N} \varphi(y - q_j(t)).
\]

(1.35)

Finally, we show (1.26). First, we recall that

\[
(\nabla \times A)^k = \varepsilon^{k\ell m} \partial_{\ell} A^m,
\]

where \( \varepsilon^{k\ell m} \) is the completely antisymmetric Levi-Civita tensor. And therefore

\[
\frac{\delta}{\delta A^i(t, y)} (\nabla \times A(t, x))^k = \varepsilon^{k\ell m} \partial_{\ell} \delta(x - y) = \varepsilon^{k\ell i} \partial_{\ell} \delta(x - y)
\]

(1.36)

\[
\frac{\delta L}{\delta A^i(t, y)} = -\int d^3x (-\partial_t A - \nabla \phi) \delta(x - y) = -E(t, y),
\]

(1.38)

\[
\partial_t \frac{\delta L}{\delta A^i(t, y)} = -\partial_t E(t, y),
\]

(1.39)

\[
\frac{\delta L}{\delta A^i(t, y)} = \int d^3x \left( -(\nabla \times A)^k \varepsilon^{k\ell i} \partial_{\ell} \delta(x - y) + e \sum_{j=1}^{N} \varphi(x - q_j(t)) \dot{q}_j \delta(x - y) \right)
\]

\[
= \int d^3x \left( -\varepsilon^{i\ell k} \partial_{\ell} (\nabla \times A)^k \delta(x - y) + e \sum_{j=1}^{N} \varphi(x - q_j(t)) \dot{q}_j \delta(x - y) \right)
\]

\[
= -(\nabla \times (\nabla \times A))^i(t, y) + j^i(t, y)
\]

\[
= -(\nabla \times B)^i(t, y) + j^i(t, y).
\]

(1.40)

### 1.4 Transverse and longitudinal field components

Recall that photon has just two polarizations, whereas here we use four functions \((\phi, A)\) to describe the electromagnetic field. To obtain the physical Hamiltonian
(energy-operator) we have to eliminate the superfluous degrees of freedom. There is a general formalism of quantisation with constraints, that could be applied here \[3\], but that would take us too far from the main topic of these lectures. Instead, we will make some informed guesses and in the end check that the resulting quantum theory satisfies Maxwell equations.

To start with, it is convenient to pass to the Fourier representation:

\[
\hat{E}(k) := \frac{1}{(2\pi)^{3/2}} \int d^3x \, e^{-ikx} E(x),
\]

\[
\hat{B}(k) := \frac{1}{(2\pi)^{3/2}} \int d^3x \, e^{-ikx} B(x),
\]

\[
\hat{A}(k) := \frac{1}{(2\pi)^{3/2}} \int d^3x \, e^{-ikx} A(x).
\]

From now on we suppress the \(t\)-dependence in the notation.

**Problem:** Show that \(\hat{\nabla} \cdot \hat{E}(k) = ik \cdot \hat{E}(k)\).

Now let \(\hat{k} := k/|k|\) and \(P_k\) the corresponding projection. That is, for any vector \(v \in \mathbb{R}^3\) we have

\[
P_k v = \hat{k} (\hat{k} \cdot v).
\]

(In Dirac notation from quantum mechanics: \(P_k = |\hat{k}\rangle \langle \hat{k}|\)). Now we can decompose the fields as follows

\[
\hat{E}(k) = P_k \hat{E}(k) + (1 - P_k) \hat{E}(k) = \hat{E}_\parallel(k) + \hat{E}_\bot(k),
\]

where \(\hat{E}_\parallel(k), \hat{E}_\bot(k)\) are called the longitudinal and transverse components. \((B\) and \(A\) are decomposed analogously). Now we note

\[
\nabla \cdot B = 0 \quad \Rightarrow \quad \hat{\nabla} \cdot \hat{B} = 0 \quad \Rightarrow \quad ik \cdot \hat{B} = 0 \quad \Rightarrow \quad \hat{B}_\parallel = 0,
\]

\[
\nabla \cdot E = \rho \quad \Rightarrow \quad \hat{\nabla} \cdot \hat{E} = \hat{\rho} \quad \Rightarrow \quad ik \cdot \hat{E} = \hat{\rho} \quad \Rightarrow \quad \hat{E}_\parallel = -i\hat{\rho} \frac{k}{|k|^2}.
\]

Given the second relation, we can eliminate \(\phi\): Note that

\[
E = -\partial_t A - \nabla \phi \quad \Rightarrow \quad \hat{E} = -\partial_t \hat{A} - i k \hat{\phi} \quad \Rightarrow \quad \hat{E}_\parallel = -\partial_t \hat{A}_\parallel - i k \hat{\phi}.
\]

Therefore,

\[
k \cdot \hat{E}_\parallel = -k \cdot \partial_t \hat{A}_\parallel - i |k|^2 \hat{\phi} \quad \Rightarrow \quad -i\hat{\rho} = -k \cdot \partial_t \hat{A}_\parallel - i |k|^2 \hat{\phi},
\]

which gives

\[
\hat{\phi} = \frac{1}{|k|^2} (ik \cdot \partial_t \hat{A}_\parallel + \hat{\rho}).
\]

Moreover,

\[
\hat{E} = \hat{E}_\bot + \hat{E}_\parallel = -\partial_t \hat{A}_\bot - i\hat{\rho} \frac{k}{|k|^2},
\]

\[
\hat{B} = \hat{B}_\bot = \hat{\nabla} \times \hat{A}_\bot = ik \times \hat{A}_\bot,
\]

because \(k \times \hat{A}_\parallel\) is proportional to \(k \times k = 0\).
1.5 Lagrangian in terms of transverse and longitudinal fields

Now we come back to the Lagrangian:

$$L = \frac{1}{2} \sum_{j=1}^{N} m \dot{q}_j^2 + \int d^3x \left( \frac{1}{2} (E(x)^2 - B(x)^2) + j(x) \cdot A(x) - \rho(x) \phi(x) \right). \quad (1.53)$$

To rewrite in terms of Fourier transformed fields, we use the Plancherel identity:

$$\int d^3x f(x) g(x) = \int d^3k \tilde{f}(k) \tilde{g}(k), \quad (1.54)$$

valid for square-integrable functions. Using (1.50), (1.51) and (1.52) we get

$$L = \frac{1}{2} \sum_{j=1}^{N} m \dot{q}_j^2 + \frac{1}{2} \int d^3k \left( |\partial_t \tilde{A}_\perp|^2 - |k \times \tilde{A}_\perp|^2 + |k|^2 |\tilde{\rho}|^2 \right)$$

$$+ \int d^3k \left( \tilde{j} \cdot \tilde{A} - |k|^2 |\tilde{\rho}|^2 - i |k|^2 \tilde{\rho} k \cdot \partial_t \tilde{A}_\parallel \right). \quad (1.55)$$

For example,

$$\int d^3x |E(x)|^2 = \int d^3k |\tilde{E}(k)|^2 = \int d^3k |\partial_t \tilde{A}_\perp + i \tilde{\rho} \frac{k}{|k|^2}|^2$$

$$= \int d^3k \left( |\partial_t \tilde{A}_\perp|^2 + |k|^2 |\tilde{\rho}|^2 \right). \quad (1.56)$$

Now we make two rearrangements of $L$:

(a) We use the charge conservation law (1.15):

$$\partial_t \rho + \nabla \cdot j = 0 \quad \Rightarrow \quad \partial_t \tilde{\rho} + i \tilde{k} \cdot \tilde{j} = 0, \quad (1.57)$$

which gives

$$\tilde{j} \cdot \tilde{A} - i |k|^2 \tilde{\rho} (k \cdot \partial_t \tilde{A}_\parallel) = \tilde{j} \cdot \tilde{A}_\perp + \tilde{j} \cdot \tilde{A}_\parallel - i |k|^2 \tilde{\rho} (k \cdot \partial_t \tilde{A}_\parallel)$$

$$= \tilde{j} \cdot \tilde{A}_\perp + \tilde{j} \cdot P_k \tilde{A}_\parallel - i |k|^2 \tilde{\rho} (k \cdot \partial_t \tilde{A}_\parallel)$$

$$= \tilde{j} \cdot \tilde{A}_\perp + |k|^2 (\tilde{j} \cdot k) (k \cdot \tilde{A}_\parallel) - i |k|^2 \tilde{\rho} (k \cdot \partial_t \tilde{A}_\parallel)$$

$$= \tilde{j} \cdot \tilde{A}_\perp - i |k|^2 (\partial_t \tilde{\rho}) (k \cdot \tilde{A}_\parallel) - i |k|^2 \tilde{\rho} (k \cdot \partial_t \tilde{A}_\parallel)$$

$$= \tilde{j} \cdot \tilde{A}_\perp - i |k|^2 \partial_t (\tilde{\rho} (k \cdot \tilde{A}_\parallel)). \quad (1.58)$$

The last term gives rise to a total time derivative contribution to $L$. Such terms have no effect on the Euler-Lagrange equations, thus can be skipped. Incidentally, this step eliminates $\tilde{A}_\parallel$ from the game. It is thus natural to set $A_\parallel = 0$ in the following. This amounts to choosing the Coulomb gauge, i.e.

$$0 = \nabla \cdot A = \nabla \cdot \tilde{A} = ik \cdot A = ik \cdot \tilde{A}_\parallel, \quad (1.59)$$

which can be done without changing fields $(E, B)$. (Cf. Subsection 1.2.) Thus we are left with two degrees of freedom of the electromagnetic field described by $A_\perp$. 

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Remark 1.2. There are two independent justifications for setting $A_{\parallel} = 0$: gauge freedom and freedom to add a total time derivative to the lagrangian. This is not a coincidence\textsuperscript{2}. Consider the gauge-dependent part of the Lagrangian:

$$L_I = \int d^3x (j(t,x) \cdot A(t,x) - \rho(t,x)\phi(t,x)) \quad \text{(1.60)}$$

and make a general gauge transformation: $A(t,x) = A'(t,x) + \nabla f(t,x)$, $\phi(t,x) = \phi'(t,x) - \partial_t f(t,x)$. Then

$$L_I = \int d^3x (j(t,x) \cdot A'(t,x) - \rho(t,x)\phi'(t,x))$$
$$+ \int d^3x (-\nabla j(t,x)f(t,x) + \rho(t,x)\partial_t f(t,x))$$
$$= \int d^3x (j(t,x) \cdot A'(t,x) - \rho(t,x)\phi'(t,x))$$
$$+ \partial_t \int d^3x \rho(t,x) f(t,x). \quad \text{(1.61)}$$

Now we choose $f$ s.t. $\hat{A'}_{\parallel} = 0$ and $\hat{\phi'}(t,k) = \frac{1}{|k|^2} \hat{\rho}(t,k)$. But from (1.58) we can read off that $\hat{f}(t,k) = -i|k|^{-2}(k \cdot \hat{A}_{\parallel}(t,k))$ does the job.

(b) Now we show that the $|k|^{-2} |\hat{\rho}|^2$ contribution gives rise to the Coulomb interaction. Let us set $V(y) = (4\pi |y|)^{-1}$. Then $\hat{V}(k) = (2\pi)^{-3/2} |k|^{-2}$. Thus we have

$$V_c = \frac{1}{2} \int d^3x d^3y \rho(x)V(x-y)\rho(y) = \frac{1}{2} \int d^3x \rho(x)(V*\rho)(x)$$
$$= \frac{1}{2} \int d^3k \bar{\rho}(k)(\hat{V} * \hat{\rho})(k) = \frac{1}{2} (2\pi)^{3/2} \int d^3k \bar{\rho}(k)\hat{V}(k)\hat{\rho}(k)$$
$$= \frac{1}{2} \int d^3k |k|^{-2} |\hat{\rho}(k)|^2. \quad \text{(1.62)}$$

Since $\rho(t,x) = e \sum_{j=1}^N \varphi(x - q_j(t))$, we have

$$V_c = \frac{1}{2} e^2 \sum_{j,j'=1}^N \int d^3x d^3y \frac{1}{4\pi |x-y+q_j(t)-q_{j'}(t)|} \varphi(y). \quad \text{(1.63)}$$

In the limit of point charges, i.e. $\varphi \rightarrow \delta$ we get for $j \neq j'$ the expected contribution $(4\pi |q_j(t) - q_{j'}(t)|)^{-1}$. For $j = j'$ this limit diverges, however. (Infinite self-energy of a point charge).

\textsuperscript{2}Thanks to Vincent Beaud for pointing this out.
Given (a), (b), the Lagrangian \( 1.55 \) has the form (up to total time derivative)

\[
L = \frac{1}{2} \sum_{j=1}^{N} m\dot{q}_j^2 - V_c + \frac{1}{2} \int d^3 k (|\partial_t \hat{A}_\perp|^2 - |k \times \hat{A}_\perp|^2) + \int d^3 k (\dot{\vec{j}} \cdot \hat{A}_\perp)
\]

\[
= \frac{1}{2} \sum_{j=1}^{N} m\dot{q}_j^2 - V_c + \frac{1}{2} \int d^3 x (|\partial_t A_\perp|^2 - |\nabla \times A_\perp|^2) + \int d^3 x (j \cdot A_\perp)
\]

\( 1.64 \)

1.6 Hamiltonian

Given formula \( 1.64 \) and recalling that \( j(t, x) = e \sum_{j=1}^{N} \varphi(x - q_j(t))\dot{q}_j \), we obtain the canonical momenta:

\[
p_j = \frac{\partial L}{\partial \dot{q}_j} = m\dot{q}_j + eA_{\perp,\varphi}(q_j), \quad (1.65)
\]

\[
\Pi(y) = \frac{\delta L}{\delta A_{\perp}(y)} = \dot{A}_\perp(y) = -E_{\perp}(y). \quad (1.66)
\]

The Hamiltonian has the following form

\[
H = \sum_{j=1}^{N} p_j \dot{q}_j + \int d^3 x \Pi(x) \dot{A}_\perp(x) - L
\]

\[
= \sum_{j=1}^{N} \frac{1}{2m} (p_j - eA_{\perp,\varphi}(q_j))^2 + V_c
\]

\[
+ \frac{1}{2} \int d^3 x \left\{ (E_{\perp})^2 + (\nabla \times A_\perp)^2 \right\}, \quad (1.67)
\]

with the canonically conjugate pairs \( \{q_j, p_j\} \) and \( \{A_\perp(x), -E_{\perp}(x)\} \). To obtain this formula, it suffices to notice that

\[
\int d^3 x (j \cdot A_\perp)(x) = e \sum_{j=1}^{N} \dot{q}_j \int d^3 x \varphi(x - q_j(t)) \cdot A_\perp(x) = e \sum_{j=1}^{N} \dot{q}_j \cdot A_{\perp,\varphi}(q_j) \quad (1.68)
\]

where we made use of \( \varphi(x) = \varphi(-x) \).

1.7 Formal quantization

We now proceed to a quantum theory, which means that \( \{q_j, p_j\} \) and \( \{A_\perp(x), -E_{\perp}(x)\} \) will be operators on some Hilbert space. We first forget about the fields and quantize the particles. We require that the following commutation relations hold:

\[
[q_i^\alpha, p_j^\beta] = q_i^\alpha p_j^\beta - p_j^\beta q_i^\alpha = i\delta_{\alpha\beta}\delta_{i,j}, \quad (1.69)
\]
where we set $\hbar = 1$. Let $\mathcal{H}_p = L^2(\mathbb{R}^{3N})$ and think about distinguishable particles (otherwise we would have to restrict attention to the subspace of symmetric/antisymmetric functions). Then it is easy to see that

$$q_\alpha^i \psi(x_1, \ldots, x_N) := x_\alpha^i \psi(x_1, \ldots, x_N), \quad (1.70)$$

$$p_\beta^j \psi(x_1, \ldots, x_N) := -i\partial_{j,\beta} \psi(x_1, \ldots, x_N), \quad (1.71)$$

satisfy (1.69).

To quantize the electromagnetic field, we will look for certain quantum fields $A_\perp, -E_\perp$ i.e. functions $x \rightarrow A_\perp(x), x \rightarrow -E_\perp(x)$ with values in operators on some Hilbert space $\mathcal{F}$ (Fock space). We stress that from now on omitted time-dependence means that the fields are evaluated at $t = 0$ i.e. $A_\perp(x) = A_\perp(t = 0, x)$. We impose the following commutation relations:

$$[A_\alpha^\perp(x), -E_\beta^\perp(x')] = i\delta_\alpha^\perp_{\alpha\beta}(x - x'), \quad (1.72)$$

$$\delta_\alpha^\perp_{\alpha\beta}(x) := (2\pi)^{-3} \int d^3k e^{ikx}(\delta_{\alpha\beta} - \hat{k}_\alpha\hat{k}_\beta), \quad (1.73)$$

which should be understood in analogy to (1.69). We will not try to explain a priori why the transverse delta function $\delta_\alpha^\perp_{\alpha\beta}(x - x')$ (and not e.g. $\delta_{\alpha\beta}\delta(x - y)$) should appear on the r.h.s. (It is plausible that this has to do with the Coulomb gauge condition $\nabla A = 0$ which we use\footnote{More precisely distributions.}). Our strategy is to first find $A_\perp(x), E_\perp(x')$ which satisfy (1.72). Then, in subsection 1.9, we will check that these quantum fields satisfy the Maxwell equations, and thus the quantization prescription (1.72) was ‘correct’.

We first introduce an orthonormal basis at each $k \in \mathbb{R}^3$

$$\hat{k} = \frac{k}{|k|} \cdot \hat{e}_1(k), \hat{e}_2(k), \quad (1.74)$$

which satisfies $\hat{k} \cdot \hat{e}_\lambda(k) = 0$, $\lambda = 1, 2$, and $e_1(k) \cdot e_2(k) = 0$. Completeness of the basis can be expressed by

$$\hat{k}_\alpha^\beta \hat{k}_\alpha + \sum_{\lambda=1,2} e_\lambda^\alpha(k)e_\lambda^\beta(k) = \delta_{\alpha\beta}. \quad (1.75)$$

We introduce auxiliary distributions $(k, \lambda) \rightarrow a(k, \lambda)$ and $(k, \lambda) \rightarrow a^*(k, \lambda)$ with values in operators on the Hilbert space $\mathcal{F}$ (Fock space). They are called annihilation and creation operators, respectively, and satisfy

$$[a(k, \lambda), a^*(k', \lambda')] = \delta_{\lambda,\lambda'}\delta(k - k'), \quad (1.76)$$

$$[a(k, \lambda), a(k', \lambda')] = 0, \quad (1.77)$$

$$[a^*(k, \lambda), a^*(k', \lambda')] = 0. \quad (1.78)$$

\footnote{This form of commutation relations can be derived from a general theory of quantization with constraints, see \cite{3}.}
Their detailed definition is postponed to the next section, here we will only use these commutation relations. Now we define the fields $A_\perp(x)$, $E_\perp(x')$ via their Fourier transforms:

$$
\hat{A}_\perp(k) := \sum_{\lambda=1,2} \sqrt{\frac{1}{2\omega(k)}} (e_\lambda(k)a(k,\lambda) + e_\lambda(-k)a^*(-k,\lambda)),
$$

$$
\hat{E}_\perp(k) := \sum_{\lambda=1,2} \sqrt{\frac{\omega(k)}{2}} (ie_\lambda(k)a(k,\lambda) - ie_\lambda(-k)a^*(-k,\lambda)),
$$

where $\omega(k) = |k|$ is the dispersion relation (i.e. energy-momentum relation) of photons. We note that by definition $\hat{k} \cdot \hat{A}_\perp(k) = 0$ and $\hat{k} \cdot \hat{E}_\perp(k) = 0$ i.e. the fields are indeed transverse. In configuration space they have the form

$$
A_\perp(x) = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=1,2} \int d^3k \sqrt{\frac{1}{2\omega(k)}} e_\lambda(k)(e^{ikx}a(k,\lambda) + e^{-ikx}a^*(k,\lambda)),
$$

$$
E_\perp(x) = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=1,2} \int d^3k \sqrt{\frac{\omega(k)}{2}} e_\lambda(k)i(e^{ikx}a(k,\lambda) - e^{-ikx}a^*(k,\lambda)).
$$

In order to verify the commutation relation, \eqref{1.72}, we first compute

$$
[A_\perp^\alpha(k), \hat{E}_\perp^\beta(k')] = \sum_{\lambda,\lambda'} \sqrt{\frac{1}{2\omega(k)}} \sqrt{\frac{\omega(k')}{2}} \left[-i e^{\alpha}_{\lambda}(k)\epsilon^{\beta}_{\lambda'}(-k') [a(k,\lambda), a^*(-k',\lambda')] + e^{\alpha}_{\lambda'}(-k)\epsilon^{\beta}_{\lambda}(k') [a^*(-k,\lambda), a(k',\lambda')] \right]
$$

$$
+ \sum_{\lambda,\lambda'} \sqrt{\frac{1}{2\omega(k)}} \sqrt{\frac{\omega(k')}{2}} \left[i e^{\alpha}_{\lambda}(k)\epsilon^{\beta}_{\lambda'}(-k') [a^*(-k,\lambda), a^*(-k',\lambda')] + e^{\alpha}_{\lambda'}(-k)\epsilon^{\beta}_{\lambda}(k') [a(k,\lambda), a^*(-k',\lambda')] \right]
$$

$$
= \frac{1}{2}(-i)\delta(k+k') \left( \sum_{\lambda,\lambda'} e^{\alpha}_{\lambda}(k)\epsilon^{\beta}_{\lambda'}(k)\delta_{\lambda\lambda'} + e^{\alpha}_{\lambda'}(-k)\epsilon^{\beta}_{\lambda}(k)\delta_{\lambda\lambda'} \right)
$$

$$
= -i\delta(k+k')\left(\delta_{\alpha\beta} - \hat{k}^\alpha \hat{k}^\beta\right),
$$

where in the last step we made use of \eqref{1.75}. Now we take Fourier transforms:

$$
[A_\perp^\alpha(x), -E_\perp^\beta(x')] = i(2\pi)^{-3} \int d^3kd^3k' e^{ikx} e^{ik'x'} \delta(k+k')\left(\delta_{\alpha\beta} - \hat{k}^\alpha \hat{k}^\beta\right)
$$

$$
= i\delta^\perp_{\alpha\beta}(x-x'),
$$

which gives \eqref{1.72}. By analogous computations we also obtain

$$
[A_\perp^\alpha(x), A_\perp^\beta(x')] = 0, \quad [E_\perp^\alpha(x), E_\perp^\beta(x')] = 0.
$$
For example (EXERCISE IN CLASS):

\[
[\hat{A}_\perp^\alpha(k), \hat{A}_\perp^\beta(k')] := \sum_{\lambda,\lambda'} \sqrt{\frac{1}{2\omega(k)}} \sqrt{\frac{1}{2\omega(k')}} \left( [e^{\alpha}_\lambda(k)a(k,\lambda), e^{\beta}_\lambda(-k')a^*(-k',\lambda')] + [e^{\alpha}_\lambda(-k)a^*(-k,\lambda), e^{\beta}_\lambda(k')a(k',\lambda')] \right)
\]

\[
= \sum_{\lambda} \frac{1}{2\omega(k)} \delta(k + k') \left( e^{\alpha}_\lambda(k)e^{\beta}_\lambda(k) - e^{\alpha}_\lambda(-k)e^{\beta}_\lambda(-k) \right)
\]

\[
= \frac{1}{2\omega(k)} \delta(k + k') \left( \delta_{\alpha\beta} - \hat{k}^{\alpha}\hat{k}^{\beta} - (\delta_{\alpha\beta} - (\hat{k}^{\alpha})(-\hat{k}^{\beta})) \right)
\]

\[
= 0. \tag{1.86}
\]

The full electric field also has a longitudinal component. Recall that in the Coulomb gauge \( \hat{E}_|| = -ik\hat{\phi} \) and \( \hat{\phi} = |k|^{-2}\hat{\rho} \). Therefore

\[
E_||(x) = \frac{1}{(2\pi)^{3/2}} \int d^3 k \ e^{ikx}(-i\hat{\rho}(k)) \frac{k}{|k|^2}, \quad \rho(x) = e \sum_{j=1}^{N} \varphi(x - q_j). \tag{1.87}
\]

We have

\[
\hat{\rho}(k) = e(2\pi)^{-3/2} \sum_{j=1}^{N} \int d^3 x \ e^{-ikx} \varphi(x - q_j)
\]

\[
= e \sum_{j=1}^{N} e^{-ikq_j} (2\pi)^{-3/2} \int d^3 x \ e^{-ikx} \varphi(x) = e \sum_{j=1}^{N} e^{-ikq_j} \hat{\varphi}(k). \tag{1.88}
\]

Therefore

\[
E_||(x) = -ie \frac{1}{(2\pi)^{3/2}} \sum_{j=1}^{N} \int d^3 k \ e^{ik(x-q_j)} \hat{\varphi}(k) \frac{k}{|k|^2}, \tag{1.89}
\]

is an operator on \( \mathcal{H}_p = L^2(\mathbb{R}^{3N}) \). The full electric field \( E(x) \) is a distribution with values in \( \mathcal{H} := \mathcal{H}_p \otimes \mathcal{F} \):

\[
E(x) = E_||(x) \otimes 1 + 1 \otimes E_\perp(x). \tag{1.90}
\]

For future reference, we also state the formula for the magnetic field

\[
B(x) = \nabla \times A_\perp(x)
\]

\[
= \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=1,2} \int d^3 k \ \sqrt{\frac{1}{2\omega(k)}} \left( ik \times e_\lambda(k) \right) \left( e^{ikx} a(k,\lambda) - e^{-ikx} a^*(k,\lambda) \right).
\]

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1.8 Formal quantum Hamiltonian

Recall the classical expression (1.67) for the Hamiltonian: (For clarity we denote now classical quantities by ‘check’):

\[
\hat{H} = \sum_{j=1}^{N} \frac{1}{2m}(\ddot{p}_j - e\tilde{A}_{\perp,\varphi}(\ddot{q}_j))^2 + \dot{V}_c + \hat{H}_f, \tag{1.91}
\]

\[
\hat{H}_f := \frac{1}{2} \int d^3 x \left\{ (\dot{E}_{\perp}(x))^2 + \frac{1}{2}(\nabla \times \tilde{A}_{\perp}(x))^2 \right\}. \tag{1.92}
\]

We want to replace the classical quantities in the expression above by their quantum counterparts and skip ‘checks’. Two problems may appear: First, multiplication of quantum quantities (operators) is not commutative - this may lead to ambiguities. Second, quantum fields are distributions, thus expressions like \((\dot{E}_{\perp}(x))^2\) may be problematic. Therefore we proceed in small steps:

(a) \(\ddot{q}_j, \ddot{p}_j \to q_j, p_j\), defined in (1.70), (1.71), which act on \(\mathcal{H}_p := L^2(\mathbb{R}^{3N})\).

(b) \(\tilde{A}_{\perp}(x), \tilde{E}_{\perp}(x) \to A_{\perp}(x), E_{\perp}(x)\), defined in (1.81), (1.82), are distributions with values in operators on \(\mathcal{F}\).

(c) \(\tilde{A}_{\perp,\varphi}(x) \to A_{\perp,\varphi}(x)\) is a function with values in operators on \(\mathcal{F}\). Explicitly, we have

\[
\tilde{A}_{\perp,\varphi}(k) = \tilde{A}(\varphi)(k) = (2\pi)^{3/2}\tilde{\varphi}(k)\tilde{A}(k). \tag{1.93}
\]

Thus making use of \(\varphi(x) = \varphi(-x)\) and therefore \(\tilde{\varphi}(k) = \tilde{\varphi}(-k)\), we get

\[
A_{\perp,\varphi}(x) = \sum_{\lambda=1,2} \int d^3 k \frac{\tilde{\varphi}(k)}{\sqrt{2\omega(k)}} e_\lambda(k) \left( e^{ikx} a(k, \lambda) + e^{-ikx} a^*(k, \lambda) \right). \tag{1.94}
\]

(d) \(\tilde{A}_{\perp,\varphi}(q_j) \to A_{\perp,\varphi}(q_j)\) involves both particle and photon degrees of freedom. It is an operator on \(\mathcal{H} := \mathcal{H}_p \otimes \mathcal{F}\), which is the full Hilbert space of the model. It is defined as

\[
A_{\perp,\varphi}(q_j) = \sum_{\lambda=1,2} \int d^3 k \frac{\tilde{\varphi}(k)}{\sqrt{2\omega(k)}} e_\lambda(k) \left( e^{ikq_j} a(k, \lambda) + e^{-ikq_j} a^*(k, \lambda) \right). \tag{1.95}
\]

(The symbol \(\otimes\) will be often omitted for brevity).

(e) \((\ddot{p}_j - e\tilde{A}_{\perp,\varphi}(\ddot{q}_j)) \to (p_j \otimes 1 - eA_{\perp,\varphi}(q_j))\) is an operator on \(\mathcal{H} = \mathcal{H}_p \otimes \mathcal{F}\).

(f) \((\ddot{p}_j - e\tilde{A}_{\perp,\varphi}(\ddot{q}_j))^2 \to (p_j \otimes 1 - eA_{\perp,\varphi}(q_j))^2\) is an operator on \(\mathcal{H} = \mathcal{H}_p \otimes \mathcal{F}\), but we have to be careful about ordering of operators: We have

\[
(p_j \otimes 1 - eA_{\perp,\varphi}(q_j))^2 = (p_j \otimes 1 - eA_{\perp,\varphi}(q_j))(p_j \otimes 1 - eA_{\perp,\varphi}(q_j))
\]

\[
= (p_j \otimes 1)^2 - e(p_j \otimes 1)A_{\perp,\varphi}(q_j) - eA_{\perp,\varphi}(q_j)(p_j \otimes 1) + (eA_{\perp,\varphi}(q_j))^2. \tag{1.96}
\]
But if we expanded the square on the classical side, quantization could give a priori different expressions:

\[
(p_j \otimes 1)^2 - 2e(p_j \otimes 1)A_{\perp \varphi}(q_j) + (eA_{\perp \varphi}(q_j))^2,
\]

\[(1.97)\]

\[
(p_j \otimes 1)^2 - 2eA_{\perp \varphi}(q_j)(p_j \otimes 1) + (eA_{\perp \varphi}(q_j))^2.
\]

\[(1.98)\]

One could argue that (1.96) is correct, because it gives a manifestly symmetric operator (recall that given two operators \(A, B\) s.t. \(A = A^*\) and \(B = B^*\) then \((AB)^* = BA\) and therefore \((AB + BA)^* = (AB + BA)\)). But fortunately all the expressions above turn out to be equal: Note that by

\[
[q_\alpha^\beta, p_\beta^\alpha] = i\delta_{\alpha\beta}\delta_{i,j}
\]

\[(1.99)\]

Therefore,

\[
[(p_\alpha^\beta \otimes 1), A_{\perp \varphi}^\alpha(q_j)]
\]

\[
= \sum_{\lambda=1,2} \int d^3k \frac{\hat{\varphi}(k)}{\sqrt{2\omega(k)}} e_\alpha^\alpha(k)(k_\alpha^\alpha e^{ikq_j} \otimes a(k, \lambda) - k_\alpha^\alpha e^{-ikq_j} \otimes a^*(k, \lambda))
\]

\[(1.100)\]

where summation over \(\alpha\) is understood and in the last step we made use of transversality: \(e_\alpha^\alpha(k)k_\alpha^\alpha = 0\). For completeness, we still have to show (1.99):

(EXERCISE IN CLASS)

\[
[p_\alpha^\alpha, e^{ikq_j}] = k_\alpha^\alpha e^{ikq_j}, \quad [p_\alpha^\beta, e^{-ikq_j}] = -k_\alpha^\alpha e^{-ikq_j}.
\]

\[(1.101)\]

Now we consider a function \(f(\eta) = e^{ikq_j}p_\alpha^\alpha e^{-ikq_j}\) and compute

\[
\partial_\eta f(\eta) = ik_\beta^\alpha e^{ikq_j}[p_\beta^\alpha, p_\alpha^\beta e^{-ikq_j}] = ik_\beta^\alpha e^{ikq_j}i\delta_{\alpha\beta}e^{-ikq_j} = -k_\alpha^\alpha.
\]

\[(1.102)\]

Therefore \(f(\eta) = f(0) - k_\alpha^\alpha \eta = p_\alpha^\alpha - k_\alpha^\alpha \eta\), in particular \(f(1) = e^{ikq_j}p_\alpha^\alpha e^{-ikq_j} = p_\alpha^\alpha - k_\alpha^\alpha\). Substituting this to (1.101) we conclude the proof of the first identity in (1.99). Second identity follows by conjugation.

\[(g) \int d^3x (E_\perp(x))^2 \rightarrow \int d^3x : (E_\perp(x))^2 :\text{ is an operator on } \mathcal{F} \text{. The 'Wick ordering': ... : of a quadratic expression in } a^*(k, \lambda) \text{ and } a(k, \lambda) \text{ means that all creation operators should be shifted to the left of all the annihilation operators. This operation is needed to make sense of squaring distributions. We} \]
have by the Plancherel theorem:

$$\int d^3x : (E(x))^2 := \int d^3k : \hat{E}_\perp(k)^* \cdot \hat{E}_\perp(k) :$$

$$= \sum_{\lambda, \lambda'} \int d^3k \frac{|k|}{2} (e_\lambda(k)a^*(k, \lambda) - e_\lambda(-k)a(-k, \lambda)) \cdot (e_{\lambda'}(k)a(k, \lambda') - e_{\lambda'}(-k)a^*(-k, \lambda')) :$$

$$= \sum_{\lambda, \lambda'} \int d^3k \frac{|k|}{2} \left( e_\lambda(k) \cdot e_{\lambda'}(k)a^*(k, \lambda)a(k, \lambda') + e_\lambda(-k) \cdot e_{\lambda'}(-k)a^*(-k, \lambda)a(-k, \lambda') \right. - e_\lambda(k) \cdot e_{\lambda'}(-k)a^*(k, \lambda)a^*(-k, \lambda') - e_\lambda(-k) \cdot e_{\lambda'}(k)a(-k, \lambda)a(k, \lambda') \left) \right)$$

$$= \sum_{\lambda} \int d^3k |k|a^*(k, \lambda)a(k, \lambda)$$

$$- \frac{1}{2} \sum_{\lambda, \lambda'} \int d^3k |k| \left( e_\lambda(k) \cdot e_{\lambda'}(-k)a^*(k, \lambda)a^*(-k, \lambda') + h.c. \right). \tag{1.103}$$

Without Wick ordering we would have in addition infinite ‘vacuum energy’:

$$\sum_{\lambda, \lambda'} \int d^3k \frac{|k|}{2} e_\lambda(-k) \cdot e_{\lambda'}(-k)[a(-k, \lambda), a^*(-k, \lambda')]$$

$$= \sum_{\lambda, \lambda'} \int d^3k \frac{|k|}{2} e_\lambda(-k) \cdot e_{\lambda'}(-k)\delta(-k + k) = \infty \tag{1.104}$$

Using Wick ordering one can prove many nice things. For example

$$\delta(k - k') =: \delta(k - k') := [a(k), a^*(k')] := (a(k)a^*(k') - a^*(k')a(k)) := 0 \tag{1.105}$$

By integrating we even get 1 = 0. Therefore it is sometimes useful to have a more precise definition: Let $$\Omega$$ be the vacuum vector which satisfies $$a(k)\Omega = 0$$ and $$\eta_n \rightarrow \delta$$ be a smooth delta-approximating sequence. We have

$$: E_\perp(x)^2 := \lim_{n \rightarrow \infty} \left( (E \ast \eta_n)(x)E(x) - \langle \Omega, (E \ast \eta_n)(x)E(x)\Omega \rangle \right). \tag{1.106}$$

With this representation the problems above would not appear. It is also clear that one can apply the Plancherel theorem.

(h) $$\int d^3x (\nabla \times \hat{A}_\perp(x))^2 \rightarrow \int d^3x : (\nabla \times A_\perp(x))^2 :$$ Again, by the Plancherel theorem

$$\int d^3x : (\nabla \times A_\perp(x))^2 := \int d^3k : (k \times \hat{A}_\perp(k))^* \cdot (k \times \hat{A}_\perp(k)) : \tag{1.107}$$
Now we recall that \( a(b \times c) = b(c \times a) \) and \( a \times (b \times c) = b(a \cdot c) - c(a \cdot b) \). Therefore, since \( \hat{A}_\perp(k), \hat{A}_\perp(k') \) commute,

\[
: (k \times \hat{A}_\perp(k^*) \cdot (k \times \hat{A}_\perp(k)) : = k : (\hat{A}_\perp(k) \times (k \times \hat{A}_\perp(k^*)) : = k \cdot (k : (\hat{A}_\perp(k)^* \cdot \hat{A}_\perp(k)) : - : \hat{A}_\perp(k)^* (k \cdot \hat{A}_\perp(k)) : ) = |k|^2 : (\hat{A}_\perp(k)^* \cdot \hat{A}_\perp(k)) :, \tag{1.108}
\]

where we made use of the transversality condition. Thus we have

\[
\int d^3k |k|^2 : (\hat{A}_\perp(k)^* \cdot \hat{A}_\perp(k)) : = \frac{1}{2} \int d^3k |k| \sum_{\lambda,\lambda'} : (e_\lambda(k) a^*(k, \lambda) + e_\lambda(-k) a(-k, \lambda)) \cdot (e_{\lambda'}(k) a(k, \lambda') + e_{\lambda'}(-k) a^*(-k, \lambda')) : = \sum_{\lambda} \int d^3k |k| a^*(k, \lambda) a(k, \lambda) + \frac{1}{2} \sum_{\lambda,\lambda'} \int d^3k |k| (e_\lambda(k) \cdot e_{\lambda'}(-k) a^*(k, \lambda) a^*(-k, \lambda') + h.c.) \tag{1.109}
\]

(i) \( \hat{H}_f \to H_f := \sum_{\lambda} \int d^3k |k| a^*(k, \lambda) a(k, \lambda) \) which is an operator on \( \mathcal{F} \). (Follows from (g) (h) above). The corresponding contribution to the Hamiltonian is \( 1 \otimes H_f \) as an operator on \( \mathcal{H} = \mathcal{H}_p \otimes \mathcal{F} \).

Altogether, the quantum Hamiltonian has the form:

\[
H = \sum_{j=1}^{N} \frac{1}{2m} (p_j \otimes 1 - eA_{\perp,\varphi}(q_j))^2 + V_c(q) \otimes 1 + 1 \otimes H_f, \tag{1.110}
\]

Usually we will just skip \( \otimes \) and write like in the classical case

\[
H = \sum_{j=1}^{N} \frac{1}{2m} (p_j - eA_{\perp,\varphi}(q_j))^2 + V_c(q) + H_f. \tag{1.111}
\]

### 1.9 Quantum Maxwell-Newton equations

Given the Hamiltonian, we can define the time-evolution of the quantum fields in the Heisenberg picture

\[
E(t, x) := e^{itH} ((1 \otimes E_\perp(x)) + (E_\parallel(x) \otimes 1)) e^{-itH}, \tag{1.112}
\]

\[
A(t, x) := e^{itH} (1 \otimes A_\perp(x)) e^{-itH}, \tag{1.113}
\]

\[
B(t, x) := e^{itH} (1 \otimes (\nabla \times A_\perp(x))) e^{-itH}, \tag{1.114}
\]

\[
\rho(t, x) := e^{itH} (\rho(x) \otimes 1) e^{-itH}. \tag{1.115}
\]
There is still one quantity missing, namely the current: Recall the classical current:

\[ j(t, x) = e^{N \sum_{j=1}^N \varphi(x - q_j(t))q_j(t)}. \]  

(EXERCISE IN CLASS: QUANTIZE). We first define the velocity operator

\[ v_j^\alpha := \frac{1}{m} (p_j \otimes 1 - eA_{\perp,\varphi}(q_j))^\alpha. \]  

and then set

\[ j(x) := \frac{1}{2} \sum_j (ev_j(\varphi(q_j - x) \otimes 1) + h.c.), \]

\[ j(t, x) = e^{itH} j(x)e^{-itH}. \]

This is a symmetric quantization of the classical current (1.116).

1.9.1 Verification of \( \nabla \cdot B = 0 \) and \( \nabla \cdot E = \rho \)

We have

\[
\nabla B(t, x) = e^{itH} \left( 1 \otimes \nabla \cdot (\nabla \times A_{\perp}(x)) \right) e^{-itH} = 0,
\]

\[
\nabla E(t, x) = e^{itH} (\nabla E_{\parallel}(x) \otimes 1) e^{-itH} = e^{itH} (\rho(x) \otimes 1) e^{-itH} = \rho(t, x),
\]

where we made use of the relation:

\[
E_{\parallel}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k e^{ikx} (-i\hat{\rho}(k)) \frac{k}{|k|^2}.
\]

1.9.2 Verification of \( \partial_t B = -\nabla \times E \)

Now we verify the dynamical equations. We have

\[
\partial_t B(t, x) = e^{itH} i[H, (1 \otimes (\nabla \times A_{\perp}(x))) e^{-itH}.
\]

Thus we have to verify commutators with the Hamiltonian. We have

\[
[p_j \otimes 1, 1 \otimes (\nabla \times A_{\perp}(x))] = 0,
\]

\[
[A_{\perp,\varphi}(q_j), 1 \otimes (\nabla \times A_{\perp}(x))]
\]

\[= (\nabla \times) \int d^3y (\varphi(q_j - y) \otimes 1)(1 \otimes [A_{\perp}(y), A_{\perp}(x)]) = 0,\]

where we made use of (1.85). Therefore

\[ [(p_j \otimes 1 - eA_{\perp,\varphi}(q_j))^2, (\nabla \times A_{\perp}(x))] = 0. \]
Now we note (POSSIBLE EXERCISE IN CLASS):

\[
[H_f, \hat{A}_\perp(k)] = \sum_{\lambda,\lambda'} \int d^3k' |k'| \left[ a^*(k', \lambda') a(k', \lambda'), \frac{1}{\sqrt{2|k|}} \left( e_\lambda(k) a(k, \lambda) + e_\lambda(-k) a^*(-k, \lambda) \right) \right]
\]

Hence

\[
i[H_f, A_\perp(x)] = -E_\perp(x), \tag{1.128}
\]

\[
i[H_f, \nabla \times A_\perp(x)] = -\nabla \times E_\perp(x) = -\nabla \times E(x), \tag{1.129}
\]

and we get

\[
\partial_t B(t, x) = -\nabla \times E(t, x). \tag{1.130}
\]

### 1.9.3 Verification of \( \partial_t E = \nabla \times B - j \)

We write

\[
\partial_t E(t, x) = e^{itH} i[H, (1 \otimes E_\perp(x))] e^{-itH} + e^{itH} i[H, (E_\parallel(x) \otimes 1)] e^{-itH} \tag{1.131}
\]

We consider first the commutator involving \( E_\perp \). We have

\[
[p_j \otimes 1, 1 \otimes E_\perp(x)] = \int d^3y \varphi(q_j - y) \otimes 1 (1 \otimes [A_{\perp, \alpha}^\alpha(y), E_\perp^{\beta}(x)])
\]

\[
= -i \int d^3y \varphi(q_j - y) \delta^{\perp}_{\alpha\beta}(y - x) \otimes 1. \tag{1.133}
\]

Consequently,

\[
[(p_j \otimes 1 - eA_{\perp, \alpha}(q_j))^2, 1 \otimes E_\perp^{\beta}(x)]
\]

\[
= -e(p_j \otimes 1 - eA_{\perp, \alpha}(q_j))^\alpha [A_{\perp, \alpha}^\alpha(q_j), 1 \otimes E_\perp^{\beta}(x)]
\]

\[
- e[A_{\perp, \alpha}^\alpha(q_j), 1 \otimes E_\perp^{\beta}(x)] (p_j \otimes 1 - eA_{\perp, \alpha}(q_j))^\alpha
\]

\[
= iemv_j^\alpha \left( \int d^3y \varphi(q_j - y) \delta^{\perp}_{\alpha\beta}(y - x) \otimes 1 \right) + h.c., \tag{1.134}
\]

with summation over \( \alpha \) and definition of the velocity operator

\[
v_j^\alpha := \frac{1}{m} (p_j \otimes 1 - eA_{\perp, \alpha}(q_j))^\alpha. \tag{1.135}
\]
Now we look at the commutator with the photon energy:

\[ [H_f, \hat{E}_\perp(k)] \]

\[ = \sum_{\lambda, \lambda'} \int d^3k' |k'| \left[ a^*(k', \lambda') a(k', \lambda'), \sqrt{\frac{|k|}{2}} \left( ie_\lambda(k) a(k, \lambda) - ie_\lambda(-k) a^*(-k, \lambda) \right) \right] \]

\[ = \sum_{\lambda, \lambda'} \int d^3k' \frac{|k'|^{3/2}}{\sqrt{2}} \left( - ie_\lambda(k) \delta(k - k') a(k', \lambda') \delta_{\lambda\lambda'} - ie_\lambda(-k) \delta(k + k') a^*(k', \lambda') \delta_{\lambda\lambda'} \right) \]

\[ = (-i) \sum_{\lambda} \frac{|k|^{3/2}}{\sqrt{2}} \left( e_\lambda(k) a(k, \lambda) + e_\lambda(-k) a^*(-k, \lambda) \right). \quad (1.136) \]

We show that the r.h.s. equals \((-i) \nabla \times B\). We have

\[ \nabla \times B(k) = \nabla \times \nabla \hat{A}_\perp(k) = -(k \times (k \times \hat{A}_\perp(k))) \]

\[ = -k(k \cdot \hat{A}_\perp(k)) + k^2 \hat{A}_\perp(k) = k^2 \hat{A}_\perp(k) \]

\[ = \sum_{\lambda} |k|^{3/2} \left( e_\lambda(k) a(k, \lambda) + e_\lambda(-k) a^*(-k, \lambda) \right). \quad (1.137) \]

Thus indeed we get

\[ i[H_f, E_\perp(x)] = \nabla \times B(x). \quad (1.138) \]

Altogether, the result for the transversal part is

\[ i[H, 1 \otimes E_\perp^\beta(x)] = (\nabla \times B)^\beta(x) \]

\[ - \frac{1}{2} \sum_j \left( ev_j^\alpha \left( \int d^3y \varphi(q_j - y) \delta_{\alpha\beta}^\perp(y - x) \otimes 1 \right) + h.c. \right). \quad (1.139) \]

Now we look at the longitudinal part: We recall

\[ E_\parallel(x) = -ie \frac{1}{(2\pi)^{3/2}} \sum_{j=1}^N \int d^3k e^{i k(x - q_j)} \hat{\varphi}(k) \frac{k}{|k|^2}. \quad (1.140) \]

Since we know that \([p_j^\alpha, e^{-i k q_j}] = -k^\alpha e^{-i k q_j}\) we have

\[ [p_j^\alpha, E_\parallel^\beta(x)] = ie \frac{1}{(2\pi)^{3/2}} \int d^3k e^{i k(x - q_j)} \hat{\varphi}(k) \frac{k^\alpha k^\beta}{|k|^2}. \quad (1.141) \]

To compute the last expression, we use \( \hat{f} \hat{g} = (2\pi)^{-3/2} \hat{f} \ast \hat{g} \) with \( f(k) = e^{-i k q_j} \hat{\varphi}(k) \) and \( g(k) = k^\alpha k^\beta / |k|^2 \) and

\[ \hat{f}(x) = \varphi(x - q_j), \quad (1.142) \]

\[ \hat{g}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k e^{i k x} k^\alpha k^\beta / |k|^2 = (2\pi)^{3/2} (\delta_{\alpha\beta} \delta(x) - \delta_{\alpha\beta}^\perp(x)). \quad (1.143) \]
Therefore,

\[ [p_j^\alpha, E_\parallel^\beta(x)] = (ie) \int d^3y \varphi(q_j - y)(\delta_{\alpha\beta}\delta(y - x) - \delta^\perp_{\alpha\beta}(y - x)), \quad (1.144) \]

where we made use of symmetry of all the integrated functions. Since \( E_\parallel^\beta(x) \) depends only on \( q \), we have

\[ [A_{\perp,\varphi}(q_j), E_\parallel(x) \otimes 1] = 0. \quad (1.145) \]

Consequently,

\[ i[(p_j \otimes 1 - eA_{\perp,\varphi}(q_j))^2, E_\parallel(\otimes 1)] = (mv_j^\alpha(i[p_j^\alpha, E_\parallel^\beta(x)] \otimes 1) + h.c.) \]

\[ = (-e) \left( mv_j^\alpha(\int d^3y \varphi(q_j - y)(\delta_{\alpha\beta}\delta(y - x) - \delta^\perp_{\alpha\beta}(y - x)) \otimes 1) + h.c. \right) \quad (1.146) \]

Since \([1 \otimes H_f, E_\parallel^\beta(x) \otimes 1] = 0\), we get

\[ i[H, E_\parallel^\beta(x) \otimes 1] = -\frac{1}{2} \sum_j \left( ev_j^\alpha(\int d^3y \varphi(q_j - y)(\delta_{\alpha\beta}\delta(y - x) - \delta^\perp_{\alpha\beta}(y - x)) \otimes 1) + h.c. \right). \quad (1.147) \]

Thus, together with \( (1.139) \), we obtain

\[ i[H, E^\beta(x)] = (\nabla \times B)^\beta(x) - \frac{1}{2} \sum_j \left( ev_j^\alpha(\int d^3y \varphi(q_j - y)vq_j^\alpha\delta(y - x) \otimes 1) + h.c. \right) \]

\[ = (\nabla \times B)^\beta(x) - \frac{1}{2} \sum_j \left( ev_j^\beta(\varphi(q_j - x) \otimes 1) + h.c. \right) \quad (1.148) \]

Thus writing

\[ j(x) := \frac{1}{2} \sum_j \left( ev_j(\varphi(q_j - x) \otimes 1) + h.c. \right) \quad (1.149) \]

and \( j(t,x) := e^{itH}j(x)e^{-itH} \) we obtain

\[ \partial_t E(t,x) = (\nabla \times B)(t,x) - j(t,x). \quad (1.150) \]

We recall that \( (1.149) \) is a ‘symmetric’ quantization of the classical current

\[ \tilde{j}(t,x) = e \sum_{j=1}^N \varphi(x - q_j(t))\dot{q}_j(t). \quad (1.151) \]
References

