Algebraic Quantum Field Theory

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Literature:

1. R. Haag: *Local Quantum Physics*, Springer 1992/1996
2. H. Araki: *Mathematical Theory of Quantum Fields*, Oxford University Press 2000.
3. D. Buchholz: *Introduction to Algebraic QFT*, lectures, University of Goettingen, winter semester 2007. (Main source for Sections 1 and 2.5).

1 Algebraic structure of quantum theory

1.1 Quantum systems with a finite number of degrees of freedom

- *Observables* describe properties of measuring devices (possible measured values, commensurability properties).
- *States* describe properties of prepared ensembles (probability distributions of measured values, correlations between observables)

Mathematical description based on Hilbert space formalism, Hilbert space $\mathcal{H}$.

- Observables: self-adjoint operators A on $\mathcal{H}$.
- States: density matrices ρ on $\mathcal{H}$ (i.e. $\rho \geq 0$, $\mathrm{Tr}\,\rho = 1$).
- Expectation values $A, \rho \mapsto \mathrm{Tr}\rho A$.

Remark 1.1 *pure states ('optimal information')= rays $e^{i\phi}\phi \in \mathcal{H}$, $\|\phi\| = 1$ = orthogonal projections $\rho^2 = \rho$. (*Question: Why equivalent? *Express in a basis, there can be just one eigenvalue with multiplicity one).*

- *Usual framework*: fixed by specifying $\mathcal{H}$. E.g. for spin $\mathcal{H} = \mathbb{C}^2$, for particle $L^2(\mathbb{R}^3)$. *Question: What is the Hilbert space for a particle with spin?* $L^2(\mathbb{R}^3; \mathbb{C}^2)$.

- *Question:* Does every s.a. operator A correspond to some measurement? Does every density matrix ρ correspond to some ensamble which can be prepared? In general no. Superselection rules. For example, you cannot superpose two states with different charges.

- *New point of view:* Observables are primary objects (we specify the family of measuring devices). The rest of the theory follows.

1.1.1 Heisenberg algebra

Quantum Mechanics. Observables:
$Q_j, j = 1, \ldots, n$ and $P_k, k = 1, \ldots, n$.
($n = Nd$, N-number of particles, d-dimension of space).

We demand that observables form (generate) an algebra.

Definition 1.2 *The "free (polynomial) $*$-algebra $\mathcal{P}$" is a complex vector space whose basis vectors are monomials ("words") in Q_j, P_k (denoted $Q_{j_1} \ldots P_{k_1} \ldots Q_{j_n} \ldots P_{k_n}$).*

1. *Sums: Elements of $\mathcal{P}$ have the form*
$$\sum c_{j_1 \ldots k_n} Q_{j_1} \ldots P_{k_n}. \tag{1}$$

2. *Products: The product operation is defined on monomials by*
$$\begin{aligned}(Q_{j_1} \ldots P_{k_1} \ldots Q_{j_n} \ldots P_{k_n}) \cdot (Q_{j'_1} \ldots P_{k'_1} \ldots Q_{j'_n} \ldots P_{k'_n}) \\ = Q_{j_1} \ldots P_{k_1} \ldots Q_{j_n} \ldots P_{k_n} Q_{j'_1} \ldots P_{k'_1} \ldots Q_{j'_n} \ldots P_{k'_n}\end{aligned}$$

3. *Adjoints: $Q_j^* = Q_j$, $P_k^* = P_k$,*
$$\left(\sum c_{j_1 \ldots k_n} Q_{j_1} \ldots P_{k_n} \right)^* = \sum \overline{c_{j_1 \ldots k_n}} P_{k_n} \ldots Q_{j_1}. \tag{2}$$

4. *Unit:* 1.

The operations $(+, \cdot, ^*)$ are subject to standard rules (associativity, distributivity, antilinearity etc.) but not commutativity.

- Quantum Mechanics requires the following *relations*:
$$[Q_j, Q_k] = [P_j, P_k] = 0, \quad \big([Q_j, P_k] - i\delta_{j,k} 1\big) = 0. \tag{3}$$

- Consider a two-sided ideal $\mathcal{J}$ generated by all linear combinations of
$$A[Q_j, Q_k]B, \quad A[P_j, P_k]B, \quad A\big([Q_j, P_k] - i\delta_{j,k} 1\big)B \tag{4}$$
for all $A, B \in \mathcal{P}$.

Definition 1.3 *Quotient $\mathcal{P} \backslash \mathcal{J}$ is again a $*$-algebra, since $\mathcal{J}$ is a two-sided ideal and $\mathcal{J}^* = \mathcal{J}$. We will call it "Heisenberg algebra". This is the free algebra 'modulo relations' (3).*

1.1.2 Weyl algebra

The elements of polynomial algebra are intrinsically unbounded (values of position and momentum can be arbitrarily large). This causes technical problems. A way out is to consider their bounded functions. For $z = u + iv \in \mathbb{C}^n$ we would like to set $W(z) \approx \exp(i \sum_k (u_k P_k + v_k Q_k))$. We cannot do it directly, because exp is undefined for 'symbols' P_k, Q_k. But we can consider abstract symbols $W(z)$ satisfying the expected relations keeping in mind the formal Baker-Campbell-Hausdorff (BCH) relation. The BCH formula gives

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]} \tag{5}$$

We have $z = u + iv, z' = u' + iv'$, $W(z) = e^A$, $W(z') = e^B$, $A = i(uP + vQ)$, $B = i(u'P + v'Q)$ and $[Q, P] = i$. Thus we have

$$[A, B] = (-1)[uP + vQ, u'P + v'Q] = (-1)(ivu' - iuv') = i(uv' - vu'). \tag{6}$$

On the other hand:

$$\mathrm{Im}\langle z, z'\rangle = \mathrm{Im}\langle u + iv, u' + iv'\rangle = \mathrm{Im}(-ivu' + iuv') = uv' - vu'. \tag{7}$$

Hence

$$W(z)W(z') = e^{\frac{i}{2}\mathrm{Im}\langle z,z'\rangle} W(z + z'). \tag{8}$$

Definition 1.4 *The (pre-)Weyl algebra $\mathcal{W}$ is the free polynomial $*$-algebra generated by the symbols $W(z)$, $z \in \mathbb{C}^n$ modulo the relations*

$$W(z)W(z') - e^{\frac{i}{2}\mathrm{Im}\langle z|z'\rangle} W(z + z') = 0, \quad W(z)^* - W(-z) = 0, \tag{9}$$

where $\langle z|z'\rangle = \sum_k \bar{z}_k z'_k$ is the canonical scalar product in $\mathbb{C}^n$.

The Weyl algebra has the following properties:

1. We have $W(0) = 1$ (by the uniqueness of unity).
2. By the above $W(z)W(z)^* = W(z)^*W(z) = 1$ i.e. Weyl operators are unitary.
3. We have

$$\Big(\sum_z a_z W(z)\Big)\Big(\sum_{z'} b_{z'} W(z')\Big) = \sum_{z,z'} a_z b_{z'} e^{\frac{i}{2}\mathrm{Im}\langle z,z'\rangle} W(z + z'). \tag{10}$$

Thus elements of $\mathcal{W}$ are linear combinations of Weyl operators $W(z)$.

1.1.3 Representations of the Weyl algebra

Definition 1.5 *A $*$-representation $\pi : \mathcal{W} \mapsto B(\mathcal{H})$ is a homomorphism i.e. a map which preserves the algebraic structure. That is for $W, W_1, W_2 \in \mathcal{W}$:*

1. linearity $\pi(c_1 W_1 + c_2 W_2) = c_1 \pi(W_1) + c_2 \pi(W_2)$,
2. multiplicativity $\pi(W_1 W_2) = \pi(W_1)\pi(W_2)$,
3. symmetry $\pi(W^*) = \pi(W)^*$.

If in addition $\pi(1) = I$, we say that the representation is unital. (In these lectures we consider unital representations unless specified otherwise).

Example 1.6 *Let $\mathcal{H}_1 = L^2(\mathbb{R}^n)$ with scalar products $\langle f, g \rangle = \int d^n x\, \overline{f}(x) g(x)$. One defines*

$$\big(\pi_1(W(z))f\big)(x) = e^{\frac{i}{2}uv} e^{ivx} f(x+u), \quad z = u + iv. \tag{11}$$

(Note that for $u = 0$ $\pi_1(W(z)$ is a multiplication operator and for $v = 0$ it is a shift). This is Schrödinger representation in configuration space.

Remark 1.7 *Heuristics: Recall that $W(z) = e^{(i\sum_k (u_k P_k + v_k Q_k))}$ and Baker-Campbell-Hausdorff*

$$\begin{aligned}(e^{i(uP+vQ)} f)(x) &= e^{\frac{i}{2}uv} (e^{ivQ} e^{i\sum uP} f)(x) &(12)\\ &= e^{\frac{i}{2}uv} e^{ivx} (e^{iuP} f)(x) = e^{\frac{i}{2}uv} e^{ivx} (f)(x+u) &(13)\end{aligned}$$

For the last step note $(e^{iuP} f)(x) = (e^{iu\frac{1}{i}\partial_x} f)(x) = (\sum_n \frac{u^n}{n!} \partial_x^n f)(x) = f(x+u)$.

Example 1.8 *Let $\mathcal{H}_2 = L^2(\mathbb{R}^n)$ with scalar products $\langle f, g \rangle = \int d^n x\, \overline{f}(x) g(x)$. One defines*

$$\big(\pi_2(W(z))f\big)(x) = e^{-\frac{i}{2}uv} e^{iux} f(x-v), \quad z = u + iv. \tag{14}$$

This is Schrödinger representation in momentum space.

Relation between $(\pi_1, \mathcal{H}_1)$, $(\pi_2, \mathcal{H}_2)$ is provided by the Fourier transform

$$(\mathcal{F}f)(y) := (2\pi)^{-n/2} \int d^n x\, e^{-ixy} f(x), \tag{15}$$

$$(\mathcal{F}^{-1}f)(y) := (2\pi)^{-n/2} \int d^n x\, e^{ixy} f(x). \tag{16}$$

$\mathcal{F}$ is isometric, i.e. $\langle \mathcal{F}f, \mathcal{F}f \rangle = \langle f, f \rangle$, (Plancherel theorem) and invertible (Fourier theorem). Hence it is unitary. We have

$$\pi_2(W) = \mathcal{F}\pi_1(W)\mathcal{F}^{-1}, \quad W \in \mathcal{W}. \tag{17}$$

Definition 1.9 *Let* $(\pi_a, \mathcal{H}_a)$, $(\pi_b, \mathcal{H}_b)$ *be two representations. If there exists an invertible isometry* $U : \mathcal{H}_a \to \mathcal{H}_b$ *(a unitary) s.t.*

$$\pi_b(\,\cdot\,) = U\pi_a(\,\cdot\,)U^{-1} \tag{18}$$

the two representations are said to be (unitarily) equivalent (denoted $(\pi_a, \mathcal{H}_a) \simeq (\pi_b, \mathcal{H}_b)$*). As we will see, equivalent representations describe the same set of states.*

Is any representation of $\mathcal{W}$ unitarily equivalent to the Schrödinger representation π_1? Certainly not, because we can form direct sums e.g. $\pi = \pi_1 \oplus \pi_1$ is not unitarily equivalent to π_1. We have to restrict attention to representations which cannot be decomposed into "smaller" ones.

Definition 1.10 *Irreducibility of representations: We say that a closed subspace* $\mathcal{K} \subset \mathcal{H}$ *is invariant (under the action of* $\pi(\mathcal{W})$*) if* $\pi(\mathcal{W})\mathcal{K} \subset \mathcal{K}$*. We say that a representation of* $(\pi, \mathcal{H})$ *of* $\mathcal{W}$ *is irreducible, if the only closed invariant subspaces are* $\mathcal{H}$ *and* $\{0\}$*.*

Remark 1.11 *The Schroedinger representation* π_1 *is irreducible (Homework).*

Lemma 1.12 *Irreducibility of* $(\pi, \mathcal{H})$ *is equivalent to any of the two conditions below:*

1. *For any non-zero* $\Psi \in \mathcal{H}$

$$\overline{\{\pi(W)\Psi \,|\, W \in \mathcal{W}\}} = \mathcal{H} \tag{19}$$

(i.e. if every non-zero vector is cyclic).

2. *Given* $A \in B(\mathcal{H})$,

$$[A, \pi(W)] = 0 \quad \text{for all} \quad W \in \mathcal{W} \tag{20}$$

implies that $A \in \mathbb{C}I$ *("Schur lemma")*
(i.e. the commutant of $\pi(\mathcal{W})$ *is trivial).*

Remark 1.13 *Recall that the commutant of* $\pi(\mathcal{W})$ *is defined as*

$$\pi(\mathcal{W})' = \{\, A \in B(\mathcal{H}) \,|[A, \pi(W)] = 0 \ \text{for all}\ W \in \mathcal{W}\}. \tag{21}$$

Proof. For complete proof see e.g. Proposition 2.3.8 in [1]. We will show here only that 1. $\Rightarrow$ 2.: By contradiction, we assume that there is $A \notin \mathbb{C}I$ in $\pi(\mathcal{W})'$. If $A \in \pi(\mathcal{W})'$ then also $A^* \in \pi(\mathcal{W})'$ hence also s.a. operators $\frac{A+A^*}{2}$ and $\frac{A-A^*}{2i}$ are in $\pi(\mathcal{W})'$. Thus, we can in fact assume that there is a s.a. operator $B \in \pi(\mathcal{W})'$, $B \notin \mathbb{C}1$. Then also bounded Borel functions of B are in $\pi(\mathcal{W})'$. In particular characteristic functions $\chi_\Delta(B)$, $\Delta \subset \mathbb{R}$ (spectral projections of B) are in $\pi(\mathcal{W})'$. Since $B \notin \mathbb{C}1$, we can find $0 \neq \chi_\Delta(B) \neq I$. Let $\Psi \in \operatorname{Ran}\chi_\Delta(B)$ i.e. $\Psi = \chi_\Delta(B)\Psi$. Then for any $W \in \mathcal{W}$

$$\pi(W)\Psi = \pi(W)\chi_\Delta(B)\Psi = \chi_\Delta(B)\pi(W)\Psi, \tag{22}$$

hence Ψ cannot be cyclic because $\chi_\Delta(B)$ projects on a subspace which is strictly smaller than $\mathcal{H}$. $\Box$

Question: Are any two irreducible representations of the Weyl algebra unitarily equivalent?

Answer: In general, no. After excluding pathologies yes.

Example 1.14 *Let $\mathcal{H}_3$ be a non-separable Hilbert space with a basis $e_p, p \in \mathbb{R}^n$. Elements of $\mathcal{H}_3$:*

$$f = \sum_p c_p e_p, \quad \text{with} \quad \sum_p |c_p|^2 < \infty \tag{23}$$

(i.e. all $c_p = 0$ apart from some countable set). $\langle f|f'\rangle = \sum_p \bar{c}_p c'_p$. We define

$$\pi_3(W(z))e_p = e^{-\frac{i}{2}uv} e^{iup} e_{p+v}. \tag{24}$$

This representation is irreducible but not unitarily equivalent to $(\pi_1, \mathcal{H}_1) \simeq (\pi_2, \mathcal{H}_2)$ because $\mathcal{H}_{1,2}$ and $\mathcal{H}_3$ have different dimension.

Criterion: Representation $(\pi, \mathcal{H})$ of $\mathcal{W}$ is of "physical interest" if for any $f \in \mathcal{H}$ the expectation values

$$z \mapsto \langle f, \pi(W(z))f\rangle \tag{25}$$

depend continuously on z.

Physical meaning of the Criterion: Set $v = 0$. Then $u \mapsto \pi(W(u))$ is an n-parameter unitary representation of translations on $\mathcal{H}$. Hence, by the Criterion and Stone's theorem

$$\pi(W(u)) = e^{i(u_1 P_{\pi,1} + \cdots + u_n P_{\pi,n})}, \tag{26}$$

where $P_{\pi,i}$ is a family of commuting s.a operators on (a domain in) $\mathcal{H}$. They can be interpreted as momentum operators in this representation. Analogously, we obtain the position operators $Q_{\pi,i}$. By taking derivatives of the Weyl relations w.r.t, u_l, v_k one obtains $[Q_{\pi,j}, P_{\pi,k}] = i\delta_{j,k}1$ on a certain domain (on which the derivatives exist).

Theorem 1.15 *(Stone-von Neumann uniqueness theorem) Any irreducible representation of $\mathcal{W}$, satisfying the Criterion, is unitarily equivalent to the Schrödinger representation.*

For a proof see Theorem 4.34 and Theorem 8.15 in [2].

Remark 1.16 *This theorem does not generalize to systems with infinitely many degrees of freedom ($n = \infty$). In particular, it does not hold in QFT. This is one reason why charges, internal ('gauge') symmetries, and groups play much more prominent role in QFT than in QM. As we will see in Section 5, they will be needed to keep track of all these inequivalent representations.*

1.1.4 States

Definition 1.17 *A state ω of a physical system is described by*

1. *specifying a representation $(\pi, \mathcal{H})$ of $\mathcal{W}$,*
2. *specifying a density matrix ρ on $\mathcal{H}$.*

Then $\omega(W) = \mathrm{Tr}\rho\pi(W)$.

Lemma 1.18 *As a map $\omega : \mathcal{W} \mapsto \mathbb{C}$, a state satisfies*

1. linearity $\omega(c_1 W_1 + c_2 W_2) = c_1\omega(W_1) + c_2\omega(W_2)$.
2. normalization $\omega(1) = 1$.
3. positivity $\omega(W^*W) \geq 0$ *for all* $W \in \mathcal{W}$.

Proof. The only non-trivial fact is positivity: Write $\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$, $p_i \geq 0$, Ψ_i orthonormal. Then, if the sum is finite, we can write

$$\begin{aligned} \omega(W^*W) &= \sum_i p_i \mathrm{Tr}(|\Psi_i\rangle\langle\Psi_i|\pi(W^*W)) \\ &= \sum_i p_i \langle\Psi_i|\pi(W^*W)\Psi_i\rangle = \sum_i p_i \|\pi(W)\Psi_i\|^2 \geq 0, \end{aligned} \tag{27}$$

by completing Ψ_i to orthonormal bases.

In the general case we can use cyclicity of the trace

$$\mathrm{Tr}\,\rho\pi(W^*W) = \mathrm{Tr}\,\rho\pi(W)^*\pi(W) = \mathrm{Tr}\,\pi(W)\rho\pi(W)^* \tag{28}$$

$$= \sum_i \sum_j p_j |\langle e_i, \pi(W)\Psi_j\rangle|^2. \tag{29}$$

The result is finite (because $\rho\pi(W^*W)$ is trace-class) and manifestly positive. □

Definition 1.19 *We say that a representation $(\pi, \mathcal{H})$ is cyclic, if $\mathcal{H}$ contains a cyclic vector Ω. (Cf. Lemma 1.12). Such representations will be denoted $(\pi, \mathcal{H}, \Omega)$. For example, any irreducible representation is cyclic.*

Theorem 1.20 *Any linear functional $\omega : \mathcal{W} \to \mathbb{C}$, which is positive and normalized, is a state in the sense of Definition 1.17 above. More precisely, it induces a unique (up to unitary equivalence) cyclic representation $(\pi, \mathcal{H}, \Omega)$ s.t.*

$$\omega(W) = \langle\Omega, \pi(W)\Omega\rangle, \quad W \in \mathcal{W}. \tag{30}$$

Proof. GNS construction (see any textbook on operator algebras, e.g. [1]). □

Lemma 1.21 *If $(\pi_1, \mathcal{H}_1) \simeq (\pi_2, \mathcal{H}_2)$ then the corresponding sets of states coincide.*

Proof. Let ρ_1 be a density matrix in representation π_1 and $W \in \mathcal{W}$. Then

$$\mathrm{Tr}\,\rho_1\pi_1(W) = \mathrm{Tr}\,\rho_1 U\pi_2(W)U^{-1} = \mathrm{Tr}\,U^{-1}\rho_1 U\pi_2(W). \tag{31}$$

Hence it does not matter if we measure W in representation π_1 on ρ_1 or in π_2 on $\rho_2 = U^{-1}\rho_1 U$.□

1.1.5 Weyl C^*-algebra

Definition 1.22 *We define a seminorm on $\mathcal{W}$:*

$$\|W\| := \sup_{\pi} \|\pi(W)\|, \quad W \in \mathcal{W}, \tag{32}$$

where the supremum extends over all cyclic representations. The completion of $\mathcal{W}/\ker\|\cdot\|$ is the Weyl C^*-algebra which we denote $\tilde{\mathcal{W}}$.

A few remarks about this definition:

1. The supremum is finite because for any representation π we have

$$\|\pi(W(z))\|^2 = \|\pi(W(z))^*\pi(W(z))\| = \|\pi(1)\| = 1 \tag{33}$$

and thus $\|\pi(W)\|$ for any $W \in \mathcal{W}$ is finite.

2. We could take supremum over all representations, but we should keep in mind that this is not a set but a class. In fact, take the direct sum of all the representations which do not have themselves as a direct summand and call this representation Π. Then we get the Russel's paradox:

$$\Pi := \bigoplus\{\pi \,|\, \pi \notin \pi\} \text{ then } \Pi \in \Pi \Leftrightarrow \Pi \notin \Pi, \tag{34}$$

where $\pi_1 \in \pi_2$ means here that π_1 is contained in π_2 as a direct summand.

3. Using the GNS theorem one can show that

$$\|W\| = \sup_{\omega} \omega(W^*W)^{1/2}. \tag{35}$$

Here the supremum extends over the *set* of states. Indeed:

$$\sup_{\omega} \omega(W^*W)^{1/2} = \sup_{(\pi,\Omega)} \langle \Omega, \pi(W^*W)\Omega\rangle^{1/2} \leq \sup_{\pi} \|\pi(W)\|. \tag{36}$$

On the other hand

$$\begin{aligned}\sup_{\pi} \|\pi(W)\| = \sup_{\pi} \sup_{\|\Psi\|=1} \|\pi(W)\Psi\| &= \sup_{\pi} \sup_{\|\Psi\|=1} \langle \Psi, \pi(W^*W)\Psi\rangle^{1/2} \\ &\leq \sup_{\omega} \omega(W^*W)^{1/2}. \end{aligned}\tag{37}$$

4. In the case of the Weyl algebra $\ker\|\cdot\| = 0$ so the seminorm (32) is actually a norm. [5]

Apart from standard properties of the norm, (32) satisfies

$$\|W_1W_2\| \leq \|W_1\|\,\|W_2\| \text{ Banach algebra property} \tag{38}$$

$$\|WW^*\| = \|W\|^2 \; C^*\text{-property} \tag{39}$$

The passage to $\tilde{W}$ is advantageous from the point of view of functional calculus: For $W \in \mathcal{W}$ we have $f(W) \in \mathcal{W}$ for polynomials f, but for more complicated functions there is no guarantee. For $W \in \tilde{\mathcal{W}}$ we have $f(W) \in \tilde{\mathcal{W}}$ for any continuous function f.

Nevertheless, in the next few subsections we will still work with the pre-Weyl algebra $\mathcal{W}$.

1.1.6 Symmetries

Postulate: Symmetry transformations are described by automorphisms (invertible homomorphisms) of $\mathcal{W}$.

Definition 1.23 *We say that a map $\alpha : \mathcal{W} \to \mathcal{W}$ is an automorphism if it is a bijection and satisfies*

1. $\alpha(c_1 W_1 + c_2 W_2) = c_1\alpha(W_1) + c_2\alpha(W_2)$
2. $\alpha(W_1 W_2) = \alpha(W_1)\alpha(W_2)$
3. $\alpha(W)^* = \alpha(W^*)$
4. $\alpha(1) = 1.$

Automorphisms of $\mathcal{W}$ form a group which we denote $\mathrm{Aut}\,\mathcal{W}$.

Example 1.24 *If $U \in \mathcal{W}$ is a unitary, then $\alpha_U(W) = UWU^{-1}$ is called an inner automorphism. Inner automorphisms form a group $\mathrm{In}\,\mathcal{W}$. For example, for $U = W(u_0)$ we have*

$$\alpha_{u_0}(W(z)) = W(u_0)W(z)W(u_0)^{-1} = e^{i\langle u_0, v\rangle}W(z) \tag{40}$$

This is translation of coordinates, as one can see in the Schroedinger representation π_1:

$$\pi_1(\alpha_{u_0}(W(z))) = e^{i\langle u_0, v\rangle}e^{i(uP+vQ)} = e^{i(uP+v(Q+u_0))}. \tag{41}$$

Similarly, for $v_0 \in \mathbb{R}^n$

$$\alpha_{iv_0}(W(z)) = W(iv_0)W(z)W(iv_0)^{-1} = e^{-i\langle v_0, u\rangle}W(z) \tag{42}$$

is a translation in momentum space.

Example 1.25 *Let $R \in SO(n)$. Then*

$$\alpha_R(W(z)) = W(Rz) \tag{43}$$

is an automorphism which is not inner. (Set $n = 3$ and let R be a rotation around the z axis by angle θ. Then, in the Schrödinger representation

$$\pi_1(\alpha_R(W(z))) = U\pi_1(W(z))U^{-1} \tag{44}$$

$U = e^{i\theta L_z}$, where $L_z = Q_x P_y - Q_y P_x$. Clearly, U is not an element of $\mathcal{W}$). Automorphisms which are not inner are called outer automorphisms. They form a set $\mathrm{Out}\mathcal{W}$ which is not a group.

As we have seen above, even if an automorphism is not inner, it can be implemented by a unitary in some given representation.

Definition 1.26 *Let* $(\pi, \mathcal{H})$ *be a representation of* $\mathcal{W}$. *Then* $\alpha \in \mathrm{Aut}\mathcal{W}$ *is said to be unitarily implementable on* $\mathcal{H}$ *if there exists some unitary* $U \in B(\mathcal{H})$ *s.t.*

$$\pi(\alpha(W)) = U\pi(W)U^{-1}, \quad W \in \mathcal{W}. \tag{45}$$

Example 1.27 *A large class of automorphisms is obtained as follows*

$$\alpha(W(z)) = c(z)W(Sz) \tag{46}$$

where $c(z) \in \mathbb{C}\backslash\{0\}$ *and* $S : \mathbb{C}^n \to \mathbb{C}^n$ *a continuous bijection. Weyl relations impose restrictions on* c, S*:*

$$c(z + z') = c(z)c(z'), \quad c(-z) = \overline{c(z)}, \quad |c(z)| = 1, \tag{47}$$

$$S(z + z') = S(z) + S(z'), \quad S(-z) = -S(z), \quad \mathrm{Im}\langle Sz, Sz'\rangle = \mathrm{Im}\langle z|z'\rangle. \tag{48}$$

The latter property means that S *is a real-linear symplectic transformation.*

For continuous c *and* S *such automorphisms are unitarily implementable in all irreducible representations satisfying the Criterion (consequence of the v.N. uniqueness theorem). See Homeworks.*

Remark 1.28 $\omega(z_1, z_2) := \mathrm{Im}\langle z|z'\rangle$ *is an example of a* symplectic form. *In general, we say that a bilinear form* ω *is symplectic if it is:*

1. *Antisymmetric:* $\omega(z_1, z_2) = -\omega(z_2, z_1)$
2. *Non-degenerate: If* $\omega(z_1, z_2) = 0$ *for all* z_2*, then* $z_1 = 0$.

1.1.7 Dynamics

Definition 1.29 *A dynamics on* $\mathcal{W}$ *is a one-parameter group of automorphisms on* $\mathcal{W}$ *i.e.* $\mathbb{R} \ni t \mapsto \alpha_t$ *s.t.* $\alpha_0 = \mathrm{id}$, $\alpha_{t+s} = \alpha_t \circ \alpha_s$.

Proposition 1.30 *Suppose that the dynamics is unitarily implemented in an irreducible representation* π *i.e. there exists a family of unitaries s.t.*

$$\pi(\alpha_t(W)) = U(t)\pi(W)U(t)^{-1}, \quad W \in \mathcal{W}. \tag{49}$$

Suppose in addition that $t \mapsto U(t)$ *continuous (in the sense of matrix elements) and differentiable (i.e. for some* $0 \neq \Psi \in \mathcal{H}$, $\partial_t U(t)\Psi$ *exists in norm).*

Then there exists a continuous group *of unitaries* $t \mapsto V(t)$ *(i.e.* $V(0) = 1$, $V(s + t) = V(s)V(t)$*) s.t.*

$$\pi(\alpha_t(W)) = V(t)\pi(W)V(t)^{-1}. \tag{50}$$

Remark 1.31 *By the Stone's theorem we have* $V(t) = e^{itH}$ *for some self-adjoint operator* H *on (a domain in)* $\mathcal{H}$ *(the Hamiltonian). Whereas* α_t *is intrinsic, the Hamiltonian is not. Its properties (spectrum etc.) depend in general on representation.*

Proof. We have $\alpha_s \circ \alpha_t = \alpha_{s+t}$. Hence

$$U(s)U(t)\pi(W)U(t)^{-1}U(s)^{-1} = U(s+t)\pi(W)U(s+t)^{-1}, \tag{51}$$
$$U(s+t)^{-1}U(s)U(t)\pi(W) = \pi(W)U(s+t)^{-1}U(s)U(t). \tag{52}$$

By irreducibility of π

$$U(s+t) = \eta(s,t)U(s)U(t), \text{ where } |\eta(s,t)| = 1. \tag{53}$$

By multiplying U by a constant phase $e^{i\phi_0}$ we can assume that $U(0) = I$, hence

$$\eta(0,t) = \eta(s,0) = 1. \tag{54}$$

Now consider a new family of unitaries $V(s) = \xi(s)U(s)$, $|\xi(s)| = 1$. We have

$$\begin{aligned} V(s+t) &= \eta'(s,t)V(s)V(t) = \xi(s+t)U(s+t) \\ &= \xi(s+t)\eta(s,t)U(s)U(t) = \xi(s+t)\eta(s,t)\xi(s)^{-1}\xi(t)^{-1}V(s)V(t). \end{aligned} \tag{55}$$

Hence

$$\eta'(s,t) = \frac{\xi(s+t)}{\xi(s)\xi(t)}\eta(s,t). \tag{56}$$

The task is to obtain $\eta'(s,t) = 1$ for all s,t for a suitable choice of ξ (depending on η). The key observation is that associativity of addition in $\mathbb{R}$ imposes a constraint on η: In fact, we can write

$$U(r+s+t) = \eta(r,s+t)U(r)U(s+t) = \eta(r,s+t)\eta(s,t)U(r)U(s)U(t), \tag{57}$$
$$U(r+s+t) = \eta(r+s,t)U(r+s)U(t) = \eta(r+s,t)\eta(r,s)U(r)U(s)U(t). \tag{58}$$

Hence we get the "cocycle relation" (cohomology theory)

$$\eta(r,s+t)\eta(s,t) = \eta(r+s,t)\eta(r,s). \tag{59}$$

Using this relation one can show that given η one can find such ξ that $\eta' = 1$. "cocycle is a coboundary" (Howework). Important intermediate step is to show, using the cocycle relation that

$$\eta(s,t) = \eta(t,s). \tag{60}$$

To express ξ as a function of η we will have to differentiate η. By assumption, there is $\Psi \in \mathcal{H}$, $\|\Psi\|$=1 s.t. $\partial_t U(t)\Psi$ exists. By (53), we have

$$\begin{aligned} \eta(s,t) &= U(t)^*U(s)^*U(s+t) = \langle \Psi, U(t)^*U(s)^*U(s+t)\Psi\rangle \\ &= \langle U(t)\Psi, U(s)^*U(s+t)\Psi\rangle. \end{aligned} \tag{61}$$

Hence $\partial_t\eta(s,t)$ exists and by (60) also $\partial_s\eta(s,t)$. $\square$

Example 1.32 Isotropic harmonic oscillator*: In the framework of the polynomial algebra $\mathcal{P}$ we have (heuristically)*

$$\begin{aligned} \alpha_t(Q_i) &= \cos(\omega_0 t)Q_i - \sin(\omega_0 t)P_i, && (62) \\ \alpha_t(P_i) &= \cos(\omega_0 t)P_i + \sin(\omega_0 t)Q_i. && (63) \end{aligned}$$

In the Weyl setting $\alpha_t(W(z)) = W(e^{it\omega_0}z)$. This defines a group of automorphisms from Example 1.27 with $S_t z = e^{it\omega_0}z$, $c(z) = 1$. (S_t is complex-linear). This dynamics is unitarily implemented in the Schrödinger representation:

$$\pi_1(\alpha_t(W)) = U(t)\pi_1(W)U(t)^{-1}, \quad W \in \mathcal{W}, \tag{64}$$

$U(t) = e^{itH}$, $H = \sum_i \left(\frac{P_i^2}{2m} + \frac{kQ_i^2}{2}\right)$, $\omega_0 = \sqrt{\frac{k}{m}}$.

Example 1.33 *Free motion in the framework of $\mathcal{P}$:*

$$\alpha_t(Q_j) = Q_j + \frac{t}{m}P_j, \tag{65}$$

$$\alpha_t(P_k) = P_k. \tag{66}$$

In the framework of $\mathcal{W}$:

$$\alpha_t(W(z)) = W(\mathrm{Re} z + (t/m + i)\mathrm{Im}\, z) \tag{67}$$

We have that $S_t(z) = \mathrm{Re} z + (t/m+i)\mathrm{Im}\, z$ is a symplectic transformation, but only real linear. This dynamics is unitarily implemented in the Schrödinger representation:

$$\pi_1(\alpha_t(W)) = U(t)\pi_1(W)U(t)^{-1}, \quad W \in \mathcal{W}, \tag{68}$$

$U(t) = e^{itH}$, $H = \sum_i \frac{P_i^2}{2m}$.

By generalizing the above discussion, one can show that dynamics governed by Hamiltonians which are quadratic in P_i, Q_j correspond to groups of automorphisms of $\mathcal{W}$. But there are many other interesting Hamiltonians, for example:

$$H = \frac{P^2}{2m} + V(Q) \tag{69}$$

where $n = 1$, $V \in C_0^\infty(\mathbb{R})_{\mathbb{R}}$ (smooth, compactly supported, real).

Theorem 1.34 *(No-go theorem) Let $H = \frac{P^2}{2m} + V(Q)$, $V \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $U(t) = e^{itH}$. Then*

$$U(t)\pi_1(W)U(t)^{-1} \in \pi_1(\widetilde{\mathcal{W}}), \quad W \in \widetilde{\mathcal{W}}, t \in \mathbb{R}. \tag{70}$$

implies that $V = 0$.

Proof. See [3]. □

Thus $\mathrm{Aut}\tilde{\mathcal{W}}$ does not contain dynamics corresponding to Hamiltonians (69). A recently proposed solution to this problem is to pass from exponentials $W(z) = e^{i(uP+vQ)}$ to resolvents $R(\lambda, z) = (i\lambda - uP - vQ)^{-1}$ and work with an algebra generated by these resolvents [4].

1.1.8 Resolvent algebra

Definition 1.35 *The pre-resolvent algebra $\mathcal{R}$ is the free polynomial $*$-algebra generated by symbols $R(\lambda, z)$, $\lambda \in \mathbb{R}\backslash\{0\}$, $z \in \mathbb{C}^n$ modulo the relations*

$$
\begin{aligned}
R(\lambda, z) - R(\mu, z) &= i(\mu - \lambda)R(\lambda, z)R(\mu, z), && (71)\\
R(\lambda, z)^* &= R(-\lambda, z), && (72)\\
[R(\lambda, z), R(\mu, z')] &= i\mathrm{Im}\langle z, z'\rangle R(\lambda, z)R(\mu, z')^2 R(\lambda, z), && (73)\\
\nu R(\nu\lambda, \nu z) &= R(\lambda, z), && (74)\\
R(\lambda, z)R(\mu, z') &= R(\lambda + \mu, z + z')(R(\lambda, z) + R(\mu, z') \\
&\quad + i\mathrm{Im}\langle z, z'\rangle R(\lambda, z)^2 R(\mu, z')), && (75)\\
R(\lambda, 0) &= \frac{1}{i\lambda}, && (76)
\end{aligned}
$$

where $\lambda, \mu, \nu \in \mathbb{R}\backslash\{0\}$ and in (75) we require $\lambda + \mu \neq 0$.

Remark 1.36 *Heuristically $R(\lambda, z) = (i\lambda - uP - vQ)^{-1}$. Relations (71), (72) encode the algebraic properties of the resolvent of some self-adjoint operator. (73) encodes the canonical commutation relations. (74), (75), (76) encode linearity of the map $(u, v) \mapsto uP + vQ$.*

Definition 1.37 *The Schrödinger representation of $\mathcal{R}$ is defined as follows: Let $(\pi_1, \mathcal{H}_1)$ be the Schrödinger representation of $\mathcal{W}$. Since it satisfies the Criterion (i.e. it is "regular") we have P_i, Q_j as self-adjoint operators on $L^2(\mathbb{R}^n)$. Thus we can define*

$$\pi_1(R(\lambda, z)) = (i\lambda - uP - vQ)^{-1}. \tag{77}$$

One can check that this prescription defines a representation of $\mathcal{R}$ which is irreducible.

Definition 1.38 *We define a seminorm on $\mathcal{R}$*

$$\|R\| = \sup_{\pi} \|\pi(R)\|, \quad R \in \mathcal{R}, \tag{78}$$

where the supremum is over all cyclic representations of $\mathcal{R}$. (A cyclic representation is a one containing a cyclic vector. In particular, irreducible representations are cyclic). The resolvent C^*-algebra $\widetilde{\mathcal{R}}$ is defined as the completion of $\mathcal{R}/\ker\|\cdot\|$.

Remark 1.39 *The supremum is finite because for any representation π we have*

$$\|\pi(R(\lambda, z))\| \leq \frac{1}{\lambda}, \quad \textit{(Homework)}. \tag{79}$$

and thus $\|\pi(R)\|$ for any $R \in \mathcal{R}$ is finite. It is not known if $\ker\|\cdot\|$ is trivial. To show that it would suffice to exhibit one representation of $\mathcal{R}$ which is faithful *(i.e. injective: $\pi(R) = 0$ implies $R = 0$). A natural candidate is the Schrödinger representation. In this case one would have to check that if*

$$\sum_{finite} c_{i_1,\dots i_n} \pi_1\big(R(\lambda_{i_1}, z_{i_1}) \cdots R(\lambda_{i_n}, z_{i_n})\big) = 0 \tag{80}$$

Then all $c_{i_1,\dots i_n} = 0$.

Definition 1.40 *A representation $(\pi, \mathcal{H})$ of $\tilde{\mathcal{R}}$ is regular if there exist self-adjoint operators P_i, Q_j on $\mathcal{H}$ s.t. for $\lambda \in \mathbb{R}\backslash\{0\}$*

$$\pi(R(\lambda, z)) = (i\lambda - uP - vQ)^{-1}. \tag{81}$$

For example, the Schrödinger representation π_1 (of $\widetilde{\mathcal{R}}$) is regular.

Fact: Any regular irreducible representation π of $\tilde{\mathcal{R}}$ is faithful [4]. Hence, the Schrödinger representation of $\widetilde{\mathcal{R}}$ is faithful. This does *not* imply however that the Schrödinger representation of $\mathcal{R}$ is faithful since we divided by $\ker \|\cdot\|$!

Proposition 1.41 *There is a one-to-one correspondence between regular representations of $\widetilde{\mathcal{R}}$ and representations of $\widetilde{\mathcal{W}}$ satisfying the Criterion. (The latter are also called "regular"). Hence, by the Stone-von Neumann uniqueness theorem, any irreducible regular representation of $\widetilde{\mathcal{R}}$ is unitarily equivalent to the Schrödinger representation.*

Proof. (Idea). Use the Laplace transformation

$$\pi(R(\lambda, z)) = -i \int_0^{\sigma\lambda} e^{-\lambda t} \pi(W(-tz))dt, \quad \sigma = \text{sgn}\lambda \tag{82}$$

to construct a regular representation of $\tilde{\mathcal{R}}$ out of a regular representation of $\tilde{\mathcal{W}}$. □

Remark 1.42 *The Laplace transform can also be useful in checking if* $\ker \|\cdot\|$ *is trivial.*

Up to now, we found no essential difference between the Weyl algebra and the resolvent algebra. An important difference is that the Weyl C^*-algebra $\widetilde{\mathcal{W}}$ is simple, i.e. it has no non-trivial two sided ideals. The resolvent C^*-algebra has many ideals. They help to accommodate interesting dynamics.

Theorem 1.43 *There is a closed two-sided ideal $\mathcal{J} \subset \widetilde{\mathcal{R}}$ s.t. in any irreducible regular representation $(\pi, \mathcal{H})$ one has $\pi(\mathcal{J}) = \mathcal{K}(\mathcal{H})$ where $\mathcal{K}(\mathcal{H})$ is the algebra of compact operators on $\mathcal{H}$.*

Remark 1.44 *We recall:*

- *A is a compact operator if it maps bounded operators into pre-compact operators. (On a separable Hilbert space if it is a norm limit of a sequence of finite rank operators).*

- *A is Hilbert-Schmidt ($A \in \mathcal{K}_2(\mathcal{H})$) if $\|A\|_2 := \mathrm{Tr}(A^*A)^{1/2} < \infty$. Hilbert-Schmidt operators are compact.*

- *A convenient way to show that an operator on $L^2(\mathbb{R}^n)$ is Hilbert-Schmidt is to study its integral kernel K, defined by the relation:*

$$(A\Psi)(p) = \int dp'\, K(p,p')\Psi(p'). \tag{83}$$

  If K is in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ then $A \in \mathcal{K}_2(L^2(\mathbb{R}^n))$ and $\|A\|_2 = \|K\|_2$.

- *For example, consider $A = f(Q)g(P)$. Its integral kernel in momentum space is determined as follows:*

$$\begin{aligned}(f(Q)g(P)\Psi)(p) &= \frac{1}{\sqrt{2\pi}}\int dp'\, e^{iQp'}(\mathcal{F}f)(p')(g(P)\Psi)(p)\\ &= \frac{1}{\sqrt{2\pi}}\int dp'\,(\mathcal{F}f)(p')(g(P)\Psi)(p-p')\\ &= \frac{1}{\sqrt{2\pi}}\int dp'\,(\mathcal{F}f)(p')g(p-p')\Psi(p-p')\\ &= \frac{1}{\sqrt{2\pi}}\int dp''\,(\mathcal{F}f)(p-p'')g(p'')\Psi(p''). \end{aligned} \tag{84}$$

  Hence the integral kernel of $f(Q)g(P)$ if $K(p,p') = (\mathcal{F}f)(p-p')g(p')$. If f, g are square-integrable, so is K.

Proof. (Idea). By the von Neumann uniqueness theorem we can assume that π is the Schrödinger representation π_1. Then it is easy to show that $\pi(\widetilde{\mathcal{R}})$ contains some compact operators: For example, set $u_i = (\underbrace{0,\ldots,1}_{i},\ldots,0)$ and $v_i = (\underbrace{0,\ldots,1}_{i},\ldots,0)$. Then the operator

$$\begin{aligned} A &:= \pi_1(R(\lambda_1, iv_1)R(\mu_1,u_1)\ldots R(\lambda_n, iv_n)R(\mu_n,u_n))\\ &= \prod_{j=1}^{n}(i\lambda_j - Q_j)^{-1}\prod_{k=1}^{n}(i\mu_j - P_j)^{-1} \end{aligned} \tag{85}$$

is Hilbert-Schmidt for all $\lambda_i, \mu_i \in \mathbb{R}\backslash\{0\}$. (This can be shown by checking that it has a square-integrabe kernel). In particular it is compact. Now it is a general fact in the theory of C^*-algebras that if the image of an irreducibe representation contains one non-zero compact operator then it contains all of them (Howework). Thus, since π_1 is faithful, we can set $\mathcal{J} = \pi_1^{-1}(\mathcal{K}(\mathcal{H}))$. This is a closed two-sided ideal in $\widetilde{\mathcal{R}}$ since $\mathcal{K}(\mathcal{H})$ is a closed two-sided ideal in $B(\mathcal{H})$. $\square$

Theorem 1.45 *Let $n = 1$, $H = P^2 + V(Q)$, where $V \in C_0(\mathbb{R})_{\mathbb{R}}$ real, continuous vanishing at infinity and $U(t) = e^{itH}$. Then*

$$U(t)\pi_1(R)U(t)^{-1} \in \pi_1(\widetilde{\mathcal{R}}), \quad \textit{for all } R \in \widetilde{\mathcal{R}}, t \in \mathbb{R}. \tag{86}$$

Remark 1.46 *Since π_1 is faithful, we can define the group of automorphisms of $\mathcal{R}$*

$$\alpha_t(R) := \pi_1^{-1}\big(U(t)\pi_1(R)U(t)^{-1}\big), \tag{87}$$

which is the dynamics governed by the Hamiltonian H.

Remark 1.47 *For simplicity, we assume that $V \in S(\mathbb{R})_{\mathbb{R}}$ and $\int dx\, V(x) = 0$. General case follows from the fact that such functions are dense in $C_0(\mathbb{R})_{\mathbb{R}}$ in supremum norm.*

Proof. Let $U_0(t) = e^{itH_0}$, where $H_0 = P^2$. Since this is a quadratic Hamiltonian, we have

$$U_0(t)\pi_1(\widetilde{\mathcal{R}})U_0(t)^{-1} \subset \pi_1(\widetilde{\mathcal{R}}). \tag{88}$$

Now we consider $\Gamma_V(t) := U(t)U_0(t)^{-1}$. It suffices to show that $\Gamma_V(t) - 1$ are compact for all $V \in C_0(\mathbb{R})_{\mathbb{R}}$ since then $\Gamma_V(t) \in \pi_1(\widetilde{\mathcal{R}})$ by Theorem 1.43 and hence

$$U(t)\pi_1(\widetilde{\mathcal{R}})U(t)^{-1} = \Gamma_V(t)U_0(t)\pi_1(\widetilde{\mathcal{R}})U_0(t)^{-1}\Gamma_V(t)^{-1} \in \pi_1(\widetilde{\mathcal{R}}), \tag{89}$$

using $\Gamma_V(t)^{-1} = \Gamma_V(t)^* \in \pi_1(\widetilde{\mathcal{R}})$.

We use the Dyson perturbation series of $\Gamma_V(t)$:

$$\Gamma_V(t) = \sum_{n=0}^{\infty} i^n \int_0^t dt_n \int_0^{t_n} dt_{n-1} \dots \int_0^{t_2} dt_1\, V_{t_1}V_{t_2}\dots V_{t_n}, \tag{90}$$

where $V_t := U_0(t)V(Q)U_0(t)^{-1}$ and the integrals are defined in the strong-operator topology, that is exist on any fixed vector. (Cf. Proposition 1.50 below).

The key observation is that $\int_0^t ds\, V_s$ are Hilbert-Schmidt. To this end compute the integral kernel K_s of V_s:

$$(K_s)(p_1,p_2) = \frac{1}{\sqrt{2\pi}} e^{ip_1^2 s}(\mathcal{F}V)(p_1 - p_2)e^{-ip_2^2 s}. \tag{91}$$

This is clearly not Hilbert-Schmidt. Now let us compute the integral kernel $\hat{K}_s$ of $\int_0^t ds\, V_s$:

$$(\hat{K}_s)(p_1,p_2) = \int_0^t ds\,(K_s)(p_1,p_2) = \frac{1}{\sqrt{2\pi}} \frac{e^{i(p_1^2-p_2^2)t} - 1}{i(p_1^2 - p_2^2)}(\mathcal{F}V)(p_1 - p_2). \tag{92}$$

This is Hilbert-Schmidt. In fact:

$$\begin{aligned} \int dp_1 dp_2\, |(\hat{K}_s)(p_1,p_2)|^2 &= c\int dq_1\, |(\mathcal{F}V)(q_1)|^2 \int dq_2 \frac{\sin^2(tq_1q_2)}{(q_1q_2)^2} \\ &= c\int dq_1\, |(\mathcal{F}V)(q_1)|^2 \frac{|t|}{|q_1|}\int dr \frac{\sin^2(r)}{r^2} \\ &= c'|t| \int dq_1\, \frac{|(\mathcal{F}V)(q_1)|^2}{|q_1|} \end{aligned} \tag{93}$$

Since $(\mathcal{F}V)(0) = 0$ we have $(\mathcal{F}V)(q_1) \leq c|q_1|$ near zero so the integral exists.

Consequently, the strong-operator continuous functions

$$\mathbb{R}^{n-1} \ni (t_2, \ldots, t_n) \mapsto \int_0^{t_2} dt_1\, V_{t_1} V_{t_2} \ldots V_{t_n} \tag{94}$$

have values in the Hilbert-Schmidt class and their Hilbert-Schmidt (HS) norms are bounded by

$$\left(c'|t_2| \int dq_1 \, \frac{|(\mathcal{F}V)(q_1)|^2}{|q_1|} \right)^{1/2} \|V\|^{n-1} \tag{95}$$

(since $\|AB\|_2 \leq \|A\|_2 \|B\|$). The integral of any strong-operator continuous HS-valued function with uniformly bounded (on compact sets) HS norm is again HS. (See Lemma 1.49 below). So each term in the Dyson expansion (apart from $n = 0$) is a Hibert-Schmidt (and therefore compact) operator. As the expansion converges uniformly in norm, the limit is also a compact operator. (Here we use that compact operators form a C^*-algebra, which is a norm closed set). So $\Gamma_V(t) - 1$ is a compact operator. □

Remark 1.48 *The resolvent algebra admits dynamics corresponding to $H = P^2 + V(Q)$. But there are other interesting Hamiltonians which are not covered e.g. $H = \sqrt{P^2 + M^2}$. So there remain open questions...*

In the above proof we used two facts, which we will now verify:

Lemma 1.49 *Let $\mathbb{R}^n \ni t \mapsto F(t) \in \mathcal{K}_2(\mathcal{H})$ be continuous in the strong operator topology and suppose that for some compact set $K \subset \mathbb{R}^n$ we have*

$$\sup_{t \in K} \|F(t)\|_2 < \infty, \tag{96}$$

where $\|F(t)\|_2 = \mathrm{Tr}(F(t)^* F(t))^{1/2}$. Then

$$\hat{F} := \int_K dt\, F(t) \tag{97}$$

is again Hilbert-Schmidt.

Proof. We have

$$\begin{aligned} \|\hat{F}\|_2^2 &= \mathrm{Tr}\, \hat{F}^* \hat{F} = |\sum_i \int_{K \times K} dt_1 dt_2 \langle e_i, F(t_1)^* F(t_2) e_i \rangle| \\ &\leq \sum_i \int_{K \times K} dt_1 dt_2 |\langle e_i, F(t_1)^* F(t_2) e_i \rangle| \\ &\leq \sum_i \int_{K \times K} dt_1 dt_2 \|F(t_1) e_i\| \, \|F(t_2) e_i\|. \end{aligned} \tag{98}$$

Since the summands/integrals are positive, I can exchange the order of integration/summation. By Cauchy-Schwarz inequality:

$$\begin{aligned} \|\hat{F}\|_2^2 &\leq \int_{K\times K} dt_1 dt_2 \big(\sum_i \|F(t_1)e_i\|^2\big)^{1/2} \big(\sum_i \|F(t_2)e_i\|^2\big)^{1/2} \\ &= \int_{K\times K} dt_1 dt_2 \, \|F(t_1)\|_2 \|F(t_2)\|_2 \\ &\leq |K|^2 \sup_{t\in K} \|F(t)\|_2^2 < \infty. \end{aligned} \tag{99}$$

Where in the last step we use the assumption (96). □

Lemma 1.50 *(Special case of Theorem 3.1.33 of [1]) Let $\mathbb{R} \ni t \mapsto U_0(t)$ be a strongly continuous group of unitaries on $\mathcal{H}$ with generator H_0 (i.e. $U_0(t) = e^{itH_0}$, above we had $H_0 = P^2$) and let V be a bounded s.a. operator on $\mathcal{H}$. Then $H_0 + V$ generates a strongly continuous group of unitaries U s.t.*

$$\begin{aligned} &U(t)\Psi = U_0(t)\Psi \\ &+ \sum_{n\geq 1} i^n \int_{0\leq t_1 \leq \cdots \leq t_n \leq t} dt_1 \dots dt_n \, U_0(t_1) V U_0(t_2 - t_1) V \cdots U_0(t_n - t_{n-1}) V U_0(t - t_n)\Psi \end{aligned} \tag{100}$$

For any $\Psi \in \mathcal{H}$. (To get the expression for $\Gamma_V(t)$ it suffices to set $\Psi = U_0(t)^{-1}\Psi'$).

Proof. Strategy: we will treat (100) as a definition of a $t \geq 0$ dependent family of operators $t \mapsto U(t)$. We will use this definition to show that it can be naturally extended to a group of unitaries parametrized by $t \in \mathbb{R}$. Then, by differentiation, we will check that its generator is $H_0 + V$. Hence, by Stone's theorem we will have $U(t) = e^{it(H_0+V)}$.

Now we give a detailed proof (NOT RELEVANT FOR EXAMINATION). Let $U^{(n)}(t)$ be the n-th term of the series of U. We have, by a change of variables,

$$U^{(0)}(t) = U_0(t), \quad U^{(n)}(t) = \int_0^t dt_1 \, U_0(t_1) i V U^{(n-1)}(t - t_1). \tag{101}$$

Iteratively, one can show that all $U^{(n)}(t)$ are well defined and strongly continuous. It is easy to check that this is a series of bounded operators which converges in norm: In fact

$$\|U^{(n)}(t)\Psi\| \leq \frac{t^n}{n!} \|V\|^n \|\Psi\|, \text{ hence } \sum_n \|U^{(n)}(t)\Psi\| < \infty. \tag{102}$$

By taking the sum of both sides of the recursion relation (101), we get

$$U(t) = U_0(t) + \int_0^t ds \, U_0(s) i V U(t - s). \tag{103}$$

Now we want to show the (semi-)group property:

$$\begin{aligned} U(t_1)U(t_2) &= U_0(t_1)U(t_2) + \int_0^{t_1} ds\, U_0(s)iVU(t_1-s)U(t_2) \\ &= U_0(t_1+t_2) + \int_0^{t_2} ds\, U_0(t_1+s)iVU(t_2-s) \\ &\quad + \int_0^{t_1} ds\, U_0(s)iVU(t_1-s)U(t_2) \\ &= U(t_1+t_2) + \int_0^{t_2} ds\, U_0(t_1+s)iVU(t_2-s) \\ &\quad + \int_0^{t_1} ds\, U_0(s)iVU(t_1-s)U(t_2) \\ &\quad - \int_0^{t_1+t_2} ds\, U_0(s)iVU(t_1+t_2-s) \end{aligned} \tag{104}$$

Now $\int_{t_1}^{t_1+t_2}$ part of the last integral cancels the $\int_0^{t_2}$ integral (change of variables). We are left with

$$U(t_1)U(t_2) - U(t_1+t_2) = \int_0^{t_1} ds\, U_0(s)iV\big(U(t_1-s)U(t_2) - U(t_1+t_2-s)\big). \tag{105}$$

Now let $U^\lambda(t)$ be defined by replacing V with λV in (100), $\lambda \in \mathbb{R}$. It is clear from (100) that the function

$$F_{t_1}(\lambda) = U^\lambda(t_1)U^\lambda(t_2) - U^\lambda(t_1+t_2) \tag{106}$$

is real-analytic. By (105) we get

$$F_{t_1}(\lambda) = \lambda \int_0^{t_1} ds\, U_0(s)iVF_{t_1-s}(\lambda). \tag{107}$$

Clearly, $F_{t_1}(0) = 0$. Using this, and differentiating the above equation w.r.t. λ at 0, we get $\partial_\lambda F_{t_1}(0) = 0$. By iterating we get that all the Taylor series coefficients of F_{t_1} at zero are zero and thus $F_{t_1}(\lambda) = 0$ by analyticity. We conclude that the semigroup property holds i.e.

$$U(t_1+t_2) = U(t_1)U(t_2). \tag{108}$$

Now we want to show that $U(t)$ are unitaries. A candidate for an inverse of $U(t)$ is $U'(t)$ defined by replacing H_0 with $H_0' := -H_0$ and V by $V' = -V$. We also set $U_0'(t) = e^{i(-H_0)t}$. Let $t_2 \geq t_1$. Then

$$\begin{aligned} U(t_1)U'(t_2) &= U_0(t_1)U'(t_2) + \int_0^{t_1} ds\, U_0(s)iVU(t_1-s)U'(t_2) \\ &= U_0(t_1-t_2) + \int_0^{t_2} ds\, U_0'(-t_1+s)iV'U'(t_2-s) \\ &\quad + \int_0^{t_1} ds\, U_0(s)iVU(t_1-s)U'(t_2) \\ &= U'(t_2-t_1) + \int_0^{t_2} ds\, U_0'(-t_1+s)iV'U'(t_2-s) \\ &\quad + \int_0^{t_1} ds\, U_0(s)iVU(t_1-s)U'(t_2) \\ &\quad - \int_0^{t_2-t_1} ds\, U_0'(s)iV'U'(t_2-t_1-s) \end{aligned} \tag{109}$$

In the last integral the part $-\int_0^{-t_1}$ combines with the second line and $-\int_{-t_1}^{-t_1+t_2}$ cancels the first line. Thus we get

$$U(t_1)U'(t_2) - U'(t_2-t_1) = \int_0^{t_1} ds\, U_0(s)iV\big(U(t_1-s)U'(t_2) - U'(t_2-(t_1-s))\big) \tag{110}$$

By an analogous argument as above we obtain

$$U(t_1)U'(t_2) = U'(t_2-t_1), \tag{111}$$

In particular, $U(t)U'(t) = 1$ and we can consistently set $U(-t) := U'(t)$ for $t \geq 0$. Moreover, it is easily seeen from (100), by a change of variables, that $U'(t) = U(t)^*$. Thus we have a group of unitaries. By Stone's theorem it has a generator which can be obtained by differentiation: Clearly we have for Ψ in the domain of H_0:

$$\partial_t|_{t=0} U_0(t)\Psi = iH_0\Psi \tag{112}$$

Now we write

$$I_t := \sum_{n\geq 1} i^n \int_0^t dt_n \int_0^{t_n} dt_{n-1} \dots \int_0^{t_2} dt_1\, V_{t_1}V_{t_2}\dots V_{t_n}U_0(t)\Psi \tag{113}$$

We have

$$\begin{aligned} \partial_t I_t &= i\sum_{n\geq 1} i^{n-1} \int_0^t dt_{n-1} \dots \int_0^{t_2} dt_1\, V_{t_1}V_{t_2}\dots V_{t_{n-1}}V_tU_0(t)\Psi \\ &+ \sum_{n\geq 1} i^n \int_0^t dt_n \int_0^{t_n} dt_{n-1} \dots \int_0^{t_2} dt_1\, V_{t_1}V_{t_2}\dots V_{t_n}U_0(t)iH_0\Psi. \end{aligned} \tag{114}$$

Taking the limit $t \to 0$ the second term tends to zero and the first term tends to zero apart from $n = 1$ (since then there are no integrals). The $n = 1$ term gives $iV\Psi$, thus, together with (112) we get that the generator of U is $H_0 + V$. $\square$

1.2 Weyl algebra for systems with infinitely many degrees of freedom

Algebraic approach is adventageous in order to perform the transition from finite to infinite systems.

- Finite systems: $\mathbb{C}^n$, $\langle \cdot, \cdot \rangle$, $\sigma(z, z') = \mathrm{Im}\langle z, z'\rangle$. Pre-Weyl algebra $\mathcal{W}$ is the free $*$-algebra generated by $W(z), z \in \mathbb{C}^n$, subject to relations

$$W(z)W(z') = e^{\frac{i}{2}\sigma(z,z')}W(z+z'), \quad W(z)^* = W(-z), \quad z \in \mathbb{C}^n. \tag{115}$$

Remark 1.51 *This form of Weyl relations corresponds to $W(z) = e^{i(uP+vQ)}$, $z = u + iv$ via BCH. If we wanted $W_{\mathrm{new}}(z) = e^{i(vP+uQ)}$, $z = u + iv$, that would lead to a minus sign in front of σ:*

$$W_{\mathrm{new}}(z)W_{\mathrm{new}}(z') = e^{-\frac{i}{2}\sigma(z,z')}W_{\mathrm{new}}(z+z') \tag{116}$$

This convention will be more convenient in the case of systems with infinitely many degrees of freedom.

- Infinite systems: infinite dimensional complex-linear space $\mathcal{D}$ with scalar product $\langle \cdot, \cdot \rangle$ (pre-Hilbert space). Define the symplectic form $\sigma(f, g) = \mathrm{Im}\,\langle f, g\rangle$, $f, g \in \mathcal{D}$. Pre-Weyl algebra $\mathcal{W}$ is the free $*$-algebra generated by $W(f), f \in \mathcal{D}$, subject to relations

$$W(f)W(g) = e^{-\frac{i}{2}\sigma(f,g)}W(f+g), \quad W(f)^* = W(-f), \quad f, g \in \mathcal{D}. \tag{117}$$

Example 1.52 *: $\mathcal{D} = S(\mathbb{R}^d)$,*

$$\langle f, g\rangle = \int d^d x\, \overline{f(x)} g(x). \tag{118}$$

Heuristics: $W(f) =$ "$e^{i\left(\varphi(\mathrm{Re}\, f)+\pi(\mathrm{Im}\, f)\right)}$", where

$$\varphi(g) := \int d^d x\, g(x)\varphi(x), \quad \pi(h) := \int d^d x\, h(x)\pi(x) \tag{119}$$

are spatial means of the quantum "field operator" $\varphi(x)$ and its "canonical conjugate momentum" $\pi(x)$. The fields φ, π satisfy formally

$$[\varphi(x), \pi(y)] = i\delta(x-y)1, \tag{120}$$

$$[\varphi(x), \varphi(y)] = [\pi(x), \pi(y)] = 0. \tag{121}$$

$\varphi(x), \pi(y)$ are not expected to be operators, but only operator valued distributions. But $\varphi(g), \pi(h)$ are expected to be operators and we have

$$[\varphi(g), \pi(h)] = i\int d^d x\, g(x)h(x)1 = i\langle \overline{g}, h\rangle 1. \tag{122}$$

Example 1.53 *: $\mathcal{D} = S(\mathbb{R}^d)$,*

$$\langle f, g\rangle = \int d^d p\, \overline{f(p)} g(p). \tag{123}$$

Here $\mathbb{R}^d$ is interpreted as momentum space.
Heuristic interpretation: $W(f) = e^{\frac{i}{\sqrt{2}}(a^*(f)+a(f))}$ where

$$a^*(f) = \int d^d p\, f(p) a^*(p), \quad a(f) = \int d^d p\, \overline{f(p)} a(p). \tag{124}$$

are creation and annihilation operators of particles with momentum in the support of f. The commutation relations are

$$[a(p), a^*(q)] = \delta(p-q)1, \tag{125}$$
$$[a(p), a(q)] = [a(p), a^*(q)] = 0. \tag{126}$$

Similarly as before a priori these are only operator valued distributions. For smeared versions we have:

$$[a(g), a^*(h)] = \int d^d p\, \overline{g(p)} h(p) 1 = \langle g, h\rangle 1. \tag{127}$$

1.2.1 Fock space

We recall the definition and basic properties of a Fock space over $\mathfrak{h} := L^2(\mathbb{R}^d, d^d x)$. We have for $n \in \mathbb{N}$

$$\otimes^n \mathfrak{h} = \mathfrak{h} \otimes \cdots \otimes \mathfrak{h} = L^2(\mathbb{R}^{nd}, d^{nd} x), \tag{128}$$
$$\otimes_s^n \mathfrak{h} = S_n(\mathfrak{h} \otimes \cdots \otimes \mathfrak{h}) = L_s^2(\mathbb{R}^{nd}, d^{nd} x), \tag{129}$$
$$\otimes_s^0 \mathfrak{h} := \mathbb{C}\Omega, \text{ where } \Omega \text{ is called the vacuum vector.} \tag{130}$$

Here S_n is the symmetrization operator defined by

$$S_n = \frac{1}{n!} \sum_{\sigma \in P_n} \sigma, \text{ where } \sigma(f_1 \otimes \cdots \otimes f_n) = f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}, \tag{131}$$

P_n is the set of all permutations and $L_s^2(\mathbb{R}^{nd}, d^{nd} x)$ is the subspace of symmetric (w.r.t. permutations of variables) square integrable functions. The (symmetric) Fock space is given by

$$\Gamma(\mathfrak{h}) := \oplus_{n\geq 0} \otimes_s^n \mathfrak{h} = \oplus_{n\geq 0} L_s^2(\mathbb{R}^{nd}, d^{nd} x). \tag{132}$$

We can write $\Psi \in \Gamma(\mathfrak{h})$ in terms of its Fock space components $\Psi = \{\Psi^{(n)}\}_{n\geq 0}$. We define a dense subspace $\Gamma_{\mathrm{fin}}(\mathfrak{h}) \subset \Gamma(\mathfrak{h})$ consisting of such Ψ that $\Psi^{(n)} = 0$ except for finitely many n. Next, we define a domain

$$D := \{\, \Psi \in \Gamma_{\mathrm{fin}}(\mathfrak{h}) \,|\, \Psi^{(n)} \in S(\mathbb{R}^{nd}) \text{ for all } n \,\}. \tag{133}$$

Now, for each $p \in \mathbb{R}^d$ we define an operator $a(p) : D \to \Gamma(\mathfrak{h})$ by

$$\begin{aligned} &(a(p)\Psi)^{(n)}(k_1, \dots, k_n) = \sqrt{n+1}\,\Psi^{(n+1)}(p, k_1, \dots, k_n), \\ &\text{In particular } a(p)\Omega = 0. \end{aligned} \tag{134}$$

Note that the adjoint of $a(p)$ is not densely defined, since formally

$$(a^*(p)\Psi)^{(n)}(k_1, \dots, k_n) = \frac{1}{\sqrt{n}} \sum_{\ell=1}^{n} \delta(p - k_\ell)\Psi^{(n-1)}(k_1, \dots, k_{\ell-1}, k_{\ell+1}, \dots, k_n) \tag{135}$$

However, $a^*(p)$ is well defined as a quadratic form on $D \times D$. Expressions

$$a(g) = \int d^d p\, a(p)\overline{g(p)}, \quad a^*(g) = \int d^d p\, a^*(p) g(p), \quad g \in S(\mathbb{R}^d), \tag{136}$$

give well-defined operators on D which can be extended to $\Gamma_{\mathrm{fin}}(\mathfrak{h})$. On this domain they act as follows

$$(a(g)\Psi)^{(n)}(k_1, \dots, k_n) = \sqrt{n+1} \int d^d p\, \overline{g}(p)\Psi^{(n+1)}(p, k_1, \dots, k_n), \tag{137}$$

$$(a^*(g)\Psi)^{(n)}(k_1, \dots, k_n) = \frac{1}{\sqrt{n}} \sum_{\ell=1}^{n} g(k_\ell)\Psi^{(n-1)}(k_1, \dots, k_{\ell-1}, k_{\ell+1}, \dots, k_n). \tag{138}$$

These expressions can be used to define $a(g), a^*(g)$ for $g \in L^2(\mathbb{R}^d)$. Since these operators leave $\Gamma_{\mathrm{fin}}(\mathfrak{h})$ invariant, one can compute on this domain:

$$[a(f), a^*(g)] = \langle f, g \rangle 1 \tag{139}$$

for $f, g \in L^2(\mathbb{R}^d)$. (Formally, this follows from $[a(p), a^*(q)] = \delta(p - q)$).

Now we are ready to define canonical fields and momenta: Let $\mu : \mathbb{R}^d \mapsto \mathbb{R}_+$ be positive, measurable function of momentum s.t. if $f \in S(\mathbb{R}^d)$ then $\mu^{1/2} f, \mu^{-1/2} f \in L^2(\mathbb{R}^d)$. (Examples: $\mu(p) = 1, \mu_m(p) = \sqrt{p^2 + m^2}$, $m \geq 0$). We set for $f, g \in S(\mathbb{R}^d)$

$$\varphi_\mu(f) := \frac{1}{\sqrt{2}}\big(a^*(\mu^{-1/2}\hat{f}) + a(\mu^{-1/2}\hat{\bar{f}})\big), \tag{140}$$

$$\pi_\mu(g) := \frac{1}{\sqrt{2}}\big(a^*(i\mu^{1/2}\hat{g}) + a(i\mu^{1/2}\hat{\bar{g}})\big), \tag{141}$$

where $\hat{f}(p) := (\mathcal{F}f)(p)$. For $\mu := \mu_m$ this is the canonical field and momentum of the free scalar relativistic quantum field theory of mass $m \geq 0$. From (139) we have

$$[\varphi_\mu(f), \pi_\mu(g)] = \frac{1}{2}\big(-\langle i\hat{\bar{g}}, \hat{f}\rangle + \langle \hat{\bar{f}}, i\hat{g}\rangle\big) = \frac{i}{2}(\langle \hat{\bar{g}}, \hat{f}\rangle + \langle \hat{\bar{f}}, \hat{g}\rangle) = i\langle \bar{f}, g\rangle, \tag{142}$$

where in the last step we made use of Plancherel theorem and

$$\langle \bar{g}, f\rangle = \int d^d x\, g(x) f(x) = \langle \bar{f}, g\rangle. \tag{143}$$

Remark 1.54 *Note that (140), (141) arise by smearing the operator-valued distributions:*

$$\varphi_{\mu_m}(x) = \frac{1}{(2\pi)^{d/2}} \int \frac{d^d k}{\sqrt{2\mu_m(k)}} \big(e^{-ikx} a^*(k) + e^{ikx} a(k)\big), \tag{144}$$

$$\pi_{\mu_m}(x) = \frac{i}{(2\pi)^{d/2}} \int d^d k \sqrt{\frac{\mu_m(k)}{2}} \big(e^{-ikx} a^*(k) - e^{ikx} a(k)\big). \tag{145}$$

Consider a unitary operator u on $\mathfrak{h}$. Then, its 'second quantization' is the following operator on the Fock space:

$$\Gamma(u)|_{\Gamma^{(n)}(\mathfrak{h})} = u \otimes \cdots \otimes u, \tag{146}$$

$$\Gamma(u)\Omega = \Omega. \tag{147}$$

where $\Gamma^{(n)}(\mathfrak{h})$ is the n-particle subspace. We have the useful relations:

$$\Gamma(u) a^*(h) \Gamma(u)^* = a^*(uh), \quad \Gamma(u) a(h) \Gamma(u)^* = a(uh). \tag{148}$$

(Note that $a^*(h)^* = a(h)$).

Consider a self-adjoint operator b on $\mathfrak{h}$. Then, its 'second quantization' is the following operator on the Fock space:

$$\mathrm{d}\Gamma(b)|_{\Gamma^{(n)}(\mathfrak{h})} = \sum_{i=1}^{n} 1 \otimes \cdots b \cdots \otimes 1, \tag{149}$$

$$\mathrm{d}\Gamma(b)\Omega = 0. \tag{150}$$

Suppose that $b = b(k)$ is a multiplication operator in momentum space on $\mathfrak{h} = L^2(\mathbb{R}^d)$. Then as an equality of quadratic forms on $D \times D$ we have

$$\mathrm{d}\Gamma(b) = \int d^d k \, b(k) a^*(k) a(k). \tag{151}$$

Moreover, suppose that $U(t) = e^{itb}$. Then

$$\Gamma(U(t)) = e^{it\mathrm{d}\Gamma(b)}. \tag{152}$$

1.2.2 Representations of the Weyl algebra

Now we are ready to define several representatons of $\mathcal{W}$ on $\Gamma(\mathfrak{h})$. We set $\mathcal{D} = S(\mathbb{R}^d)$ and $\sigma(f,g) := \mathrm{Im}\,\langle f, g\rangle$ with standard scalar product in $L^2(\mathbb{R}^d)$:

Definition 1.55 *Let μ be as above. The corresponding Fock space representation of $\mathcal{W}$ is given by*

$$\rho_\mu(W(f)) = e^{i(\varphi_\mu(\mathrm{Re}\, f) + \pi_\mu(\mathrm{Im}\, f))}. \tag{153}$$

In terms of creation and annihilation operators, we have

$$\rho_\mu(W(f)) = e^{\frac{i}{\sqrt{2}}(a^*(\hat{f}_\mu) + a(\hat{f}_\mu))}, \tag{154}$$

where $\hat{f}_\mu(p) := (\mu^{-\frac{1}{2}} \widehat{\mathrm{Re} f} + i\mu^{\frac{1}{2}} \widehat{\mathrm{Im} f})(p)$. Note that for $\mu = 1$ we have $\hat{f}_\mu(p) = \hat{f}(p)$ and thus we reproduce Examples 1.52,1.53.

Theorem 1.56 *Representations ρ_{μ_m} are faithful, irreducible and $\rho_{\mu_{m_1}}$ is not unitarily equivalent to $\rho_{\mu_{m_2}}$ for $m_1 \neq m_2$. (So Stone-von Neumann uniqueness theorem does not hold for systems with infinitely many degrees of freedom).*

Proof. See Theorem X.46 of [8].

1.2.3 Symmetries

Symmetries are represented by their automorphic action on the algebra.

Definition 1.57 *Let $(\mathcal{D}, \sigma)$ be a symplectic space. A symplectic transformation S is a linear bijection $S : \mathcal{D} \to \mathcal{D}$ s.t.*

$$\sigma(Sf, Sg) = \sigma(f, g), \quad f, g \in \mathcal{D}. \tag{155}$$

Note that S^{-1} is also a symplectic transformation.

Fact: Every symplectic transformation induces an automorphism of $\mathcal{W}$ according to the relation:

$$\alpha_S(W(f)) = W(Sf), \quad f \in \mathcal{D}. \tag{156}$$

Proposition 1.58 *Let S be a symplectic transformation s.t. also $\|\widehat{(Sf)}_\mu\| = \|\hat{f}_\mu\|$. (For $\mu = 1$ this is just unitarity of S w.r.t. the scalar product in $L^2(\mathbb{R}^d)$). Then there exists a unitary operator $U_{\mu,S}$ on $\Gamma(\mathfrak{h})$ s.t.*

$$U_{\mu,S}\rho_\mu(W)U_{\mu,S}^* = \rho_\mu(\alpha_S(W)), \quad W \in \mathcal{W}, \tag{157}$$

and $U_{\mu,S}\Omega = \Omega$. (Converse also true).

Proof. We skip the index μ. Since we know that $\rho(\mathcal{W})$ acts irreducibly on $\Gamma(\mathfrak{h})$, we have that

$$D := \{\, \rho(W)\Omega \,|\, W \in \mathcal{W} \,\} \tag{158}$$

is dense in $\Gamma(\mathfrak{h})$. On this domain we set

$$U_S\rho(W(f))\Omega = \rho(W(Sf))\Omega, \tag{159}$$

and extend by linearity to $\mathcal{W}$. By invertibility of S this has a dense range. We check that it is an isometry on this domain. For this it suffices to verify

$$\langle U_S\rho(W(f))\Omega, U_S\rho(W(g))\Omega\rangle = \langle \rho(W(f))\Omega, \rho(W(g))\Omega\rangle. \tag{160}$$

We have

$$\begin{aligned} \text{l.h.s.} &= \langle \rho(W(Sf))\Omega, \rho(W(Sg))\Omega\rangle = \langle \Omega, \rho(W(-Sf)W(Sg))\Omega\rangle \\ &= e^{\frac{i}{2}\mathrm{Im}\langle f,g\rangle}\langle \Omega, \rho(W(S(g-f)))\Omega\rangle, \end{aligned} \tag{161}$$

where we made use of the fact that S is symplectic. Let us set $h := S(g-f)$. We have

$$\begin{aligned}\langle\Omega, \rho(W(h))\Omega\rangle &= \langle\Omega, e^{\frac{i}{\sqrt{2}}(a^*(\hat{h}_\mu)+a(\hat{h}_\mu))}\Omega\rangle \\ &= e^{-\frac{1}{2}\|\hat{h}_\mu\|^2}\langle\Omega, e^{\frac{i}{\sqrt{2}}a^*(\hat{h}_\mu)}e^{\frac{i}{\sqrt{2}}a(\hat{h}_\mu)}\Omega\rangle \\ &= e^{-\frac{1}{2}\|\hat{h}_\mu\|^2} = e^{-\frac{1}{2}\|(\widehat{S(g-f)})_\mu\|^2} = e^{-\frac{1}{2}\|(\widehat{g-f})_\mu\|^2}\end{aligned}$$

where we used Baker-Campbell-Hausdorff (which can be justified by expanding exponentials into convergent series) and the additional assumption on S. This gives the r.h.s. of (160).

Now the converse: suppose that α_S is unitarily implemented in ρ_μ by a unitary $U_{\mu,S}$ s.t. $U_{\mu,S}\Omega = \Omega$. Then, in particular,

$$\begin{aligned}\langle\Omega, \rho_\mu(W(Sf))\Omega\rangle &= \langle\Omega, \rho_\mu(\alpha_S(W(f)))\Omega\rangle \\ &= \langle\Omega, U_{\mu,S}\rho_\mu(W(f))U_{\mu,S}^*\Omega\rangle = \langle\Omega, \rho_\mu(W(f))\Omega\rangle. \quad (162)\end{aligned}$$

Hence,

$$e^{-\frac{1}{2}\|(\widehat{Sf})_\mu\|^2} = e^{-\frac{1}{2}\|\widehat{f}_\mu\|^2} \quad (163)$$

which concludes the proof. □

1.2.4 Symmetries in the case $\mu = 1$ ("non-local" quantum field)

We set $\mathcal{D} = S(\mathbb{R}^d)$, $\langle f, g\rangle = \int d^dx\overline{f(x)}g(x)$, $\sigma(f,g) = \mathrm{Im}\,\langle f, g\rangle$, $m > 0$.

- Note that any unitary u on $\mathfrak{h} = L^2(\mathbb{R}^d)$, which preserves $\mathcal{D}$, gives rise to a symplectic transformation $S = u|_{\mathcal{D}}$.
- By Proposition 1.58, the automorphism induced by S is unitarily implemented on $\Gamma(\mathfrak{h})$.
- A natural candidate for the implementing unitary is $\Gamma(u)$.

1. Space translations: $(S_af)(x) = f(x-a)$ (or $\widehat{(S_af)}(p) = e^{-iak}\hat{f}(p)$). Obviously

$$\langle(S_af), (S_ag)\rangle = \int d^dx\,\overline{f(x-a)}g(x-a) = \langle f, g\rangle. \quad (164)$$

(This implies that S_a is symplectic). The implementing unitary is $U(a) = \Gamma(e^{-ipa}) = e^{-ia\mathrm{d}\Gamma(p)}$, where 'p' means the corresponding multiplication operator on $L^2(\mathbb{R}^d, d^dp)$. $P := \mathrm{d}\Gamma(p) = \int d^3k\, ka^*(k)a(k)$ can be called the 'total momentum operator'. Indeed by (148):

$$\begin{aligned}\rho_{\mu=1}(\alpha_a(W(f))) &= \rho_{\mu=1}(W(S_af)) = e^{\frac{i}{\sqrt{2}}\left(a^*(e^{-iap}\hat{f})+a(e^{-iap}\hat{f})\right)} \\ &= \Gamma(e^{-ipa})e^{\frac{i}{\sqrt{2}}\left(a^*(\hat{f})+a(\hat{f})\right)}\Gamma(e^{-ipa})^* \\ &= \Gamma(e^{-ipa})\rho_{\mu=1}(W(f))\Gamma(e^{-ipa})^*. \quad (165)\end{aligned}$$

2. Rotations: $(S_R f)(x) = f(R^{-1}x)$, $R \in SO(d)$.

$$\langle (S_R f), (S_R g)\rangle = \int d^d x\, \overline{f(R^{-1}x)} g(R^{-1}x) = \langle f, g\rangle \tag{166}$$

   The implementing unitary is $U(R) = \Gamma(u_R)$, where $(u_R g)(x) = g(R^{-1}x)$ is a unitary representation of rotations on $L^2(\mathbb{R}^d)$.

3. Time translations: $\widehat{(S_t f)}(p) = e^{it\omega(p)} \hat{f}(p)$ where $\omega(p)$ is a reasonable dispersion relation of a particle. Since we want to build a relativistic theory, we set $\omega(p) = \sqrt{p^2 + m^2}$, $m > 0$. Clearly:

$$\langle (S_t f), (S_t g)\rangle = \langle f, g\rangle. \tag{167}$$

   The implementing unitary is $U(t) = \Gamma(e^{it\omega(p)}) = e^{it\mathrm{d}\Gamma(\omega(p))}$, where

$$H := \mathrm{d}\Gamma(\omega(p)) = \int d^3k\, \omega(k) a^*(k) a(k), \tag{168}$$

   can be called the 'total energy operator' or the Hamiltonian.

   Remark 1.59 *Note that $f_t := S_{-t} f$ satisfies the Schrödinger equation:*

$$i\partial_t f_t(x) = \omega(-i\nabla) f_t. \tag{169}$$

4. Lorentz transformations

   - Minkowski spacetime: $(\mathbb{R}^{d+1}, g)$, $g = (1, -1, -1, -1)$.
   - Lorentz group: $\mathcal{L} = O(1, d) = \{\Lambda \in GL(1 + d) \,|\, \Lambda g \Lambda^T = g\}$
   - Proper Lorentz group: $\mathcal{L}_+ = SO(1, d) = \{\Lambda \in O(1, d) \,|\, \det \Lambda = 1\}$ (preserves orientation).
   - Ortochronous Lorentz group: $\mathcal{L}^\uparrow = \{\Lambda \in O(1, d) \,|\, e^T \Lambda e \geq 0\}$, where $e = (1, 0, 0, 0)$. (Preserves the direction of time)
   - Proper ortochronous Lorentz group: $\mathcal{L}_+^\uparrow = \mathcal{L}^\uparrow \cap \mathcal{L}_+$ is a symmetry group of the SM of particle physics.
   - The full Lorentz group consists of four disjoint components:

$$\mathcal{L} = \mathcal{L}_+^\uparrow \cup \mathcal{L}_+^\downarrow \cup \mathcal{L}_-^\uparrow \cup \mathcal{L}_-^\downarrow \tag{170}$$

     For $d = 3$ they can be defined using time reversal $T(t, x) = (-t, x)$ and parity $P(t, x) = (t, -x)$ transformations:

$$\mathcal{L}_+^\downarrow = TP\mathcal{L}_+^\uparrow, \quad \mathcal{L}_-^\uparrow = P\mathcal{L}_+^\uparrow, \quad \mathcal{L}_-^\downarrow = T\mathcal{L}_+^\uparrow. \tag{171}$$

Now we set

$$(S_\Lambda f)(p) = \sqrt{\frac{\omega(\Lambda^{-1}p)}{\omega(p)}} f(\Lambda^{-1}p), \quad f \in \mathcal{D}, \tag{172}$$

where $\Lambda^{-1}p$ is defined by $\Lambda^{-1}(\omega(p), p) = (\omega(\Lambda^{-1}p), \Lambda^{-1}p)$. We have

$$\langle (S_\Lambda f), (S_\Lambda f) \rangle = \langle f, g \rangle. \tag{173}$$

This can be shown (Homework) using that $\frac{d^d p}{\omega(p)}$ is a Lorentz invariant measure (unique for a fixed m and normalization, see Theorem IX.37 of [8]). Formally

$$\int d^{d+1}\tilde{p}\, \delta(\tilde{p}^2 - m^2)\theta(\tilde{p}^0)F(\tilde{p}) = \int \frac{d^d p}{2\omega(p)} F(\omega(p), p), \tag{174}$$

where $\tilde{p} = (p^0, p)$, $\tilde{p}^2 = (p^0)^2 - p^2$.

S_Λ arises by restriction to $\mathcal{D}$ of a unitary representation u_Λ of $\mathcal{L}_+^\uparrow$ acting on $\mathfrak{h} = L^2(\mathbb{R}^d)$ by formula (172). The implementing unitary is $U(\Lambda) := \Gamma(u_\Lambda)$.

5. Poincaré transformations: The (proper ortochronous) Poincaré group $\mathcal{P}_+^\uparrow = \mathbb{R}^{d+1} \rtimes \mathcal{L}_+^\uparrow$ is a set of pairs $(\tilde{x}, \Lambda)$ with the multiplication:

$$(\tilde{x}_1, \Lambda_1)(\tilde{x}_2, \Lambda_2) = (\tilde{x}_1 + \Lambda_1 \tilde{x}_2, \Lambda_1\Lambda_2). \tag{175}$$

It acts naturally on $\mathbb{R}^{d+1}$ by $(\tilde{x}, \Lambda)\tilde{y} = \Lambda\tilde{y} + \tilde{x}$. (Here we set $\tilde{x} = (t, x)$).

Note that $(\tilde{x}, \Lambda) = (\tilde{x}, I)(0, \Lambda)$. Accordingly, we define

$$S_{(\tilde{x},\Lambda)} := S_{\tilde{x}} \circ S_\Lambda = S_t \circ S_x \circ S_\Lambda \tag{176}$$

as a symplectic transformation on $\mathcal{D}$ corresponding to $(\tilde{x}, \Lambda)$. We still have to check if $(\tilde{x}, \Lambda) \mapsto \alpha_{S_{(\tilde{x},\Lambda)}}$ is a representation of a group, that is whether

$$\alpha_{S_{(\tilde{x}_1,\Lambda_1)}} \circ \alpha_{S_{(\tilde{x}_2,\Lambda_2)}} = \alpha_{S_{(\tilde{x}_1,\Lambda_1)(\tilde{x}_2,\Lambda_2)}}. \tag{177}$$

We use the fact that all these automorphisms can be implemented in the (faithful) representation $\rho_{\mu=1}$. We have

$$\begin{aligned} \rho_1(\alpha_{(\tilde{x},\Lambda)}(W(f))) &= \rho_1\big(W(S_{(\tilde{x},\Lambda)} f)\big) = \rho_1\big(W(S_t \circ S_x \circ S_\Lambda f)\big) \\ &= U(t)U(x)U(\Lambda)\rho_1\big(W(f)\big)\big(U(t)U(x)U(\Lambda)\big)^* \end{aligned} \tag{178}$$

To verify (177) it suffices to check that

$$\begin{aligned} U(\tilde{x}, \Lambda) &:= U(t)U(x)U(\Lambda) = \Gamma(e^{i\omega(p)t})\Gamma(e^{-ipx})\Gamma(u_\Lambda) \\ &= \Gamma(e^{i\omega(p)t}e^{-ipx}u_\Lambda) \end{aligned} \tag{179}$$

is a unitary representation of $\mathcal{P}_+^\uparrow$ on $\Gamma(\mathfrak{h})$. For this it suffices that

$$u_{(\tilde{x},\Lambda)} = e^{i\omega(p)t}e^{-ipx}u_\Lambda \tag{180}$$

is a unitary representation of $\mathcal{P}_+^\uparrow$ on $\mathfrak{h} = L^2(\mathbb{R}^d)$. (Homework).

Summing up, for any $m > 0$ we have a representation $P_+^\uparrow \ni (\tilde{x}, \Lambda) \mapsto \alpha^{(m,\mu=1)}_{(\tilde{x},\Lambda)}$ of the Poincaré group in Aut $\mathcal{W}$. In the representation $\rho_{\mu=1}$ automorphisms $\alpha^{(m,\mu=1)}$ are unitarily implemented by a representation $P_+^\uparrow \ni (\tilde{x}, \Lambda) \mapsto U(\tilde{x}, \Lambda)$.

Nevertheless, $(\mathcal{W}, \alpha^{(m,\mu=1)}, \rho_{\mu=1})$ does not give rise to a decent (local) relativistic QFT. Problem with causality:

- $W(f)$, supp$f \subset O$ should be an observable localized in an open bounded region $O \subset \mathbb{R}^d$ at $t = 0$.
- $\alpha_t(W(f))$ should be localized in $\{O + |\tau|\vec{n}\,,\, |\vec{n}| = 1, |\tau| \leq t\}$ in a causal theory.
- However, $\alpha_t(W(f)) = W(S_t f)$, $\widehat{(S_t f)}(p) = e^{i\omega(p)t}\hat{f}(p)$ thus $S_t f$ is not compactly supported. (Infinite propagation speed of the Schrödinger equation). In fact, since $e^{i\omega(p)t}$ is not entire analytic (cut at $p = im$), its inverse Fourier transform cannot be a compactly supported distribution (see Theorem IX.12 of [8]).

1.2.5 Symmetries in the case $\mu(p) = \sqrt{p^2 + m^2}$ (”local” quantum field)

We set $\mathcal{D} = S(\mathbb{R}^d)$, $\langle f, g\rangle = \int d^d x \overline{f}(x) g(x)$, $\sigma(f, g) = \mathrm{Im}\, \langle f, g\rangle$.

- Recall that we need symplectic transformations S s.t. $\|(Sf)_\mu\| = \|f_\mu\|$, where $\hat{f}_\mu(p) := (\mu^{-\frac{1}{2}}\widehat{\mathrm{Re} f} + i\mu^{\frac{1}{2}}\widehat{\mathrm{Im} f})(p)$.
- Note that $\|(Sf)_\mu\| = \|f_\mu\|$ does not imply in this case that S is symplectic.
- Strategy: Take the unitary u on $\mathfrak{h}$ corresponding to a given symmetry (which we know from $\mu = 1$ case) and find S s.t. $uf_\mu = (Sf)_\mu$. Then check that S is symplectic.

1. Space translations: We have $\widehat{\mathrm{Re}(S_a f)}(p) = \widehat{S_a \mathrm{Re} f}(p) = e^{-iap}\widehat{\mathrm{Re} f}(p)$ and analogously for Im. Thus $\widehat{(S_a f)}_\mu(p) = e^{-iap}\hat{f}_\mu(p)$ and therefore $\|\widehat{(S_a f)}_\mu\| = \|\hat{f}_\mu\|$ so the symmetry is unitarily implemented. The implementing unitary is the same as in the $\mu = 1$ case.
2. Rotations: Again $\widehat{\mathrm{Re}(S_R f)}(p) = \widehat{S_R \mathrm{Re} f}(p) = u_R \widehat{\mathrm{Re} f}(p)$ and analogously for Im. Since μ is rotation invariant (depends only on p^2), we have $u_R \mu(p) u_R^* = \mu(p)$ and therefore $\widehat{S_R f}_\mu(p) = (u_R \hat{f}_\mu)(p)$. Thus $\|\widehat{(S_R f)}_\mu\| = \|\hat{f}_\mu\|$ so the symmetry is unitarily implemented. The implementing unitary is the same as in the $\mu = 1$ case.
3. Time translations: First note that $\widehat{(S_t' f)}(p) = e^{it\omega(p)}\hat{f}(p)$ does NOT satisfy the additional condition. For example, for f real we have

$$\widehat{(S_t' f)}_\mu(p) = \big(\mu^{-\frac{1}{2}}(p)\cos(\omega(p)t) + i\mu^{\frac{1}{2}}(p)\sin(\omega(p)t)\big)\hat{f}(p). \tag{181}$$

The L^2 norm of this $(S'_t f)_\mu$ does depend on t. (Thus $\alpha_{S'_t}$ is not implemented in this representation by unitaries preserving the vacuum).

Instead, we consider the following group of transformations:

$$\begin{aligned} (S_t f)(x) &= (\cos(t\mu) + i\mu^{-1}\sin(t\mu))\mathrm{Re}\, f(x) \\ &\quad + i(\cos(t\mu) + i\mu\sin(t\mu))\mathrm{Im}\, f(x). \end{aligned} \tag{182}$$

Think of μ as a function of $p^2 = -\nabla_x^2$. Thus we can compute real and imaginary parts as for functions:

$$\mathrm{Re}\,(S_t f) = \cos(t\mu)\mathrm{Re}\, f - \mu\sin(t\mu)\mathrm{Im}\, f, \tag{183}$$

$$\mathrm{Im}\,(S_t f) = \mu^{-1}\sin(t\mu))\mathrm{Re}\, f + \cos(t\mu)\mathrm{Im}\, f \tag{184}$$

This is a symplectic transformation

$$\sigma(S_t f, S_t g) = \langle \mathrm{Re}\,(S_t f), \mathrm{Im}(S_t g)\rangle - (f \leftrightarrow g) \tag{185}$$

We note that terms involving $\mathrm{Re}\, f \mathrm{Re}\, g$ and $\mathrm{Im}\, f \mathrm{Im}\, g$ cancel because are invariant under $(f \leftrightarrow g)$. The remaining two terms give

$$\begin{aligned} \sigma(S_t f, S_t g) &= \langle \cos^2(t\mu)\mathrm{Re}\, f, \mathrm{Im}\, g\rangle - \langle \sin^2(t\mu)\mathrm{Im}\, f, \mathrm{Re}\, g\rangle - (f \leftrightarrow g) \\ &= \langle \mathrm{Re}\, f, \mathrm{Im}\, g\rangle - \langle \mathrm{Im}\, f, \mathrm{Re}\, g\rangle = \sigma(f, g). \end{aligned} \tag{186}$$

Next, we check $\|(S_t f)_\mu\| = \|f_\mu\|$:

$$\begin{aligned} (S_t f)_\mu &= \mu^{-\frac{1}{2}}\mathrm{Re}\,(S_t f) + i\mu^{\frac{1}{2}}\mathrm{Im}\,(S_t f) \\ &= \big(\cos(t\mu) + i\sin(t\mu)\big)\big(\mu^{-\frac{1}{2}}\mathrm{Re}\, f + i\mu^{\frac{1}{2}}\mathrm{Im}\, f\big) \\ &= e^{i\mu t} f_\mu. \end{aligned} \tag{187}$$

Hence clearly $\|(S_t f)_\mu\| = \|f_\mu\|$ and this group of automorphisms is unitarily implemented on Fock space by unitaries preserving the vacuum. They are given by $U(t) = \Gamma(e^{i\mu t}) = e^{i\mathrm{d}\Gamma(\mu(p))}$. Thus the Hamiltonian is $\mathrm{d}\Gamma(\mu(p)) = \int d^d k\, \mu(k) a^*(k) a(k)$.

Remark 1.60 *Note that $f_t := (S_t f)$ in (182) is the unique solution of the Klein-Gordon equation:*

$$(\partial_t^2 - \nabla_x^2 + m^2) f_t(x) = 0 \tag{188}$$

with the initial conditions $f_{t=0}(x) = f(x)$ and $(\partial_t f)_{t=0}(x) = (\nabla_x^2 - m^2)\mathrm{Im}\, f(x) + i\mathrm{Re}\, f(x)$. In contrast to the Schrödinger equation, KG equation has finite propagation speed: If $\mathrm{supp} f_{t=0}, \mathrm{supp}\, \partial_t f_{t=0} \subset O$ then $\mathrm{supp}\, f_t \subset \{O + |\tau|\vec{n}\,,\, |\vec{n}| = 1, |\tau| \le t\}$. This theory has good chances to be local.

4. Lorentz transformations: There exist symplectic transformations S_Λ which satisfy $\|(S_\Lambda f)_\mu\| = \|(S_\Lambda)_\mu\|$ and preserve localization (for $f \in C_0^\infty(\mathbb{R}^d)$ we have $(S_\Lambda f) \in C_0^\infty(\mathbb{R}^d)$) (Homework).

5. Poincaré transformations: For $(\tilde{x}, \Lambda) \in \mathcal{P}_+^\uparrow$ we define

$$S_{(\tilde{x},\Lambda)} := S_{\tilde{x}} \circ S_\Lambda = S_t \circ S_x \circ S_\Lambda \tag{189}$$

as a symplectic transformation on $\mathcal{D}$ corresponding to $(\tilde{x}, \Lambda)$. Obviously, $\|(S_{(\tilde{x},\Lambda)}f)_\mu\| = \|f_\mu\|$, since the individual factors satisfy this. (We note that S_x is as in the $\mu = 1$ case but S_t, S_Λ are different). The proof that $(\tilde{x}, \Lambda) \mapsto \alpha_{S_{(\tilde{x},\Lambda)}}$ is a representation of a group goes as in $\mu = 1$ case, exploiting that these automorphisms are implemented on Fock space by the same group of unitaries as in the $\mu = 1$ case.

Summing up, for any $m \geq 0$ we have a representation $P_+^\uparrow \ni (\tilde{x}, \Lambda) \mapsto \alpha^{(m)}_{(\tilde{x},\Lambda)}$ of the Poincaré group in $\mathrm{Aut}\,\mathcal{W}$. In the representation ρ_{μ_m} automorphisms $\alpha^{(m)}$ are unitarily implemented by the representation $P_+^\uparrow \ni (\tilde{x}, \Lambda) \mapsto U(\tilde{x}, \Lambda)$, the same as in the $\mu = 1$ case. Time evolution is governed by the KG equation which has finite propagation speed and Lorentz transformations act locally: we expect that $(\mathcal{W}, \alpha^{(m)}, \rho_{\mu_m})$ gives rise to a local (causal) relativistic QFT.

1.2.6 Spectrum condition (positivity of energy)

In this subsection we study the spectrum of the group of unitaries on $\Gamma(\mathfrak{h})$ implementing translations in ρ_μ, $\mu = \sqrt{p^2 + m^2}$. (The discussion below is equally valid for $\rho_{\mu=1}$ since $\mu_m(p) = \omega(p)$, hence unitaries implementing translations are the same in both representations).

$$U(t, x) = e^{iHt - iPx} = e^{i\mathrm{d}\Gamma(\mu(p))t - i\mathrm{d}\Gamma(p)x} \tag{190}$$

$H, P^1, \ldots, P^d$ is a family of commuting s.a. operators on $\Gamma(\mathfrak{h})$. Such a family has a joint spectral measure E: Let $\Delta \in \mathbb{R}^{d+1}$ be a Borel set and χ_Δ its characteristic function. Then $E(\Delta) := \chi_\Delta(H, P^1, \ldots, P^d)$. The joint spectrum of $H, P^1, \ldots, P^d$, denoted $\mathrm{Sp}\,(H, P)$ is defined as the support of E. Physically, these are the measurable values of total energy and momentum of our system.

Theorem 1.61 $\mathrm{Sp}\,(H, P) \subset \overline{V}_+$, *where* $\overline{V}_+ = \{\, (p^0, p) \in \mathbb{R}^{d+1} \,|\, p_0 \geq |p| \,\}$ *is the closed future lightcone.*

Proof. We have to show that for $\Delta \cap \overline{V}_+ = \emptyset$, Δ bounded Borel set, we have $E(\Delta) = 0$. Let $\chi_\Delta^\varepsilon \in C_0^\infty(\mathbb{R}^{d+1})$ approximate χ_Δ pointwise as $\varepsilon \to 0$. (This regularization is needed because the Fourier transform of a sharp characteristic function may not be L^1). Note that $\chi_\Delta(H, P)$ leaves $\Gamma^{(n)}(\mathfrak{h})$ invariant, thus it suffices to show that its matrix elements vanish on these subspaces. We have for

$\Psi, \Phi \in \Gamma^{(n)}(\mathfrak{h})$:

$$
\begin{aligned}
&\langle \Psi, \chi_\Delta(H,P)\Phi\rangle \\
&= \lim_{\varepsilon\to 0}\langle \Psi, \chi^\varepsilon_\Delta(H,P)\Phi\rangle \\
&= \lim_{\varepsilon\to 0}(2\pi)^{-\frac{(d+1)}{2}}\int dt\,dx\langle\Psi, U(t,x)\Phi\rangle\check{\chi}^\varepsilon{}_\Delta(t,x) \\
&= \lim_{\varepsilon\to 0}(2\pi)^{-\frac{(d+1)}{2}}\int dt\,dx\int d^{nd}p\,(\overline{\Psi}\cdot\Phi)(p_1,\dots,p_n)e^{i(\tilde{p}_1+\cdots+\tilde{p}_n)\cdot\tilde{x}}\check{\chi}^\varepsilon{}_\Delta(t,x) \\
&= \int d^{nd}p\,(\overline{\Psi}\cdot\Phi)(p_1,\dots,p_n)\chi_\Delta(\tilde{p}_1+\cdots+\tilde{p}_n), \qquad (191)
\end{aligned}
$$

where we made use of Fubini and dominated convergence. Note that $\tilde{p}=(\mu(p),p)\in\overline{V}_+$ for all $p\in\mathbb{R}^d$. Since $\overline{V}_+$ is a cone, also $\tilde{p}_1+\cdots+\tilde{p}_n\in\overline{V}_+$. Thus the last expression is zero if $\Delta\cap\overline{V}_+=\emptyset$. □

Remark 1.62 *In the proof above we used the following conventions for the Fourier transform on $\mathbb{R}^{d+1}$:*

$$\hat{f}(p^0,p) := (2\pi)^{-\frac{(d+1)}{2}}\int d^dx dt\, e^{ip^0t-ipx}f(t,x), \qquad (192)$$

$$\check{f}(t,x) := (2\pi)^{-\frac{(d+1)}{2}}\int d^dp dp^0\, e^{-ip^0t+ipx}f(p^0,p). \qquad (193)$$

A more detailed analysis of the spectrum exhibits that

- for $m>0$

$$
\begin{aligned}
\mathrm{Sp}\,(H,P) &= \{0\}\cup\{H_m\}\cup G_{2m}, \text{ where} && (194)\\
H_m &:= \{\,(p^0,p)\in\mathbb{R}^{d+1}\,|\,p^0=\sqrt{p^2+m^2}\}, && (195)\\
G_{2m} &:= \{\,(p^0,p)\in\mathbb{R}^{d+1}\,|\,p^0\geq\sqrt{p^2+(2m)^2}\}. && (196)
\end{aligned}
$$

  $\{0\}$ is a simple eigenvalue corresponding to the vacuum vector Ω. H_m is called the mass hyperboloid. The corresponding spectral subspace $E(H_m)\Gamma(\mathfrak{h})$ satisfies

$$E(H_m)\Gamma(\mathfrak{h}) = \Gamma^{(1)}(\mathfrak{h}) = \mathfrak{h}. \qquad (197)$$

  Thus it is invariant under $(\tilde{x},\Lambda)\mapsto U(\tilde{x},\Lambda)$. In fact it carries the familiar irreducible representation of $u_{(x,\Lambda)}$ given by (180). According to Wigner's definition of a particle, $E(H_m)\Gamma(\mathfrak{h})$ describes single-particle states of a particle of mass m and spin 0. G_{2m} can be called the multiparticle spectrum. (PICTURE).

- For $m=0$ we have

$$\mathrm{Sp}\,(H,P) = \overline{V_+}. \qquad (198)$$

  Again, there is a simple eigenvalue at $\{0\}$ (embedded in the multiparticle spectrum) which corresponds to the vacuum vector Ω. $H_{m=0}$ is the boundary of $\overline{V_+}$. The subspace $E(H_{m=0})\Gamma(\mathfrak{h})=\mathfrak{h}$ carries states of a single massless particle of mass zero.

1.2.7 Locality and covariance

- We fix $m \geq 0$ and use automorphisms $\alpha_{(\tilde{x},\Lambda)} := \alpha^{(m)}_{(\tilde{x},\Lambda)}$ constructed in ρ_{μ_m}.
- We set $\mathcal{D} = C_0^\infty(\mathbb{R}^d)$, $\sigma(f,g) = \mathrm{Im}\langle f, g\rangle$. (We can restrict the symplectic space from $S(\mathbb{R}^d)$ to $C_0^\infty(\mathbb{R}^d)$ because it is preserved by all the symplectic transformations we considered in $\mu = \mu_m$ case, in particular by time-translations and Lorentz transformations).
- We call $\mathcal{O}_r = \{\,(t,x) \in \mathbb{R}^{d+1} \,|\, |t| + |x| < r\}$ the standard double cone of radius r. Its base is the ball $B_r = \{x \in \mathbb{R}^d \,|\, |x| < r\}$.
- $\mathcal{W}(\mathcal{O}_r) := *\!-\mathrm{Alg}\,\{W(f)\,|\,\mathrm{supp} f \subset B_r\,\}$ is the algebra of observables[1] localized (physically measurable) in O_r.
- $\mathcal{W}(\mathcal{O}_r + \tilde{x}) := \alpha_{\tilde{x}}(\mathcal{W}(\mathcal{O}_r)) = *\!-\mathrm{Alg}\{\alpha_{\tilde{x}}(W(f))\,|\,\mathrm{supp} f \subset B_r\,\}$ is the algebra of observables localized in $\mathcal{O}_r + \tilde{x}$ where $\tilde{x} = (t,x)$.

Theorem 1.63 *Suppose that $\mathcal{O}_{r_1}$ and $\mathcal{O}_{r_2} + \tilde{x}$ are spacelike separated. Then*

$$[W_1, W_2] = 0, \quad \textit{for all} \quad W_1 \in \mathcal{W}(\mathcal{O}_{r_1}), \quad W_2 \in \mathcal{W}(\mathcal{O}_{r_2} + \tilde{x}). \tag{199}$$

Proof. By Weyl relations

$$W(f_1)W(S_{\tilde{x}}f_2) = e^{i\mathrm{Im}\,\langle S_{\tilde{x}}f_2, f_1\rangle} W(S_{\tilde{x}}f_2)W(f_1), \tag{200}$$

so we have to show that $\mathrm{Im}\,\langle S_{\tilde{x}}f_2, f_1\rangle = 0$ for $\mathrm{supp} f_1 \subset B_{r_1}$ and $\mathrm{supp}\, f_2 \subset B_{r_2}$.

First suppose that $\tilde{x} = (0,x)$, then we simply obtain that $\mathcal{O}_{r_1}$ and $\mathcal{O}_{r_2} + \tilde{x}$ are disjoint and so are B_{r_1} and $B_{r_2} + x$. Hence $\langle S_{\tilde{x}}f_2, f_1\rangle = 0$ simply due to disjointness of supports of the two functions.

Now the general case: Write $S_{\tilde{x}}f_2 = S_t(S_x f_2)$. As before $S_x f_2$ is supported in $B_{r_2} + x$. Thus, by propagation properties of solutions of the KG equation, $S_t(S_x f_2)$ is supported in $(B_{r_2} + x) + |\tau| B_1$, $|\tau| \leq t$. But spacelike separation of $\mathcal{O}_{r_1}$ and $\mathcal{O}_{r_2} + \tilde{x}$ implies that $(B_{r_2} + x) + |t| B_1$ is disjoint from B_{r_1}. So, again, $\langle S_{\tilde{x}}f_2, f_1\rangle = 0$. (PICTURE). □

- $\mathcal{W}(\Lambda O_r + \tilde{x}) := \alpha_{(\tilde{x},\Lambda)} \mathcal{W}(\mathcal{O}_r)$.
- $\mathcal{W}(\mathcal{O}) := *\!-\mathrm{Alg}\{\,\mathcal{W}(\Lambda\mathcal{O}_r + \tilde{x})\,|\, r, \Lambda, \tilde{x} \; s.t.\; \Lambda\mathcal{O}_r + \tilde{x} \subset \mathcal{O}\}$ is the algebra of observables localized in an arbitrary open bounded region $\mathcal{O} \subset \mathbb{R}^{d+1}$.
- $\widetilde{\mathcal{W}}(\mathcal{O})$ are the norm closures of $\widetilde{\mathcal{W}}(\mathcal{O})$ in the C^*-algebra $\widetilde{\mathcal{W}}$.

[1] $*$-subalgebra of $\mathcal{W}$ generated by all such $W(f)$

1.2.8 Haag-Kastler axioms

Theorem 1.64 *The net of C^*-algebras $\mathcal{O} \mapsto \widetilde{\mathcal{W}}(\mathcal{O})$, labelled by open bounded subsets $\mathcal{O} \subset \mathbb{R}^{d+1}$, satisfies:*

1. *(isotony)* $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \widetilde{\mathcal{W}}(O_1) \subset \widetilde{\mathcal{W}}(O_2)$,
2. *(locality)* $\mathcal{O}_1, \mathcal{O}_2$ *spacelike separated* $\Rightarrow [\widetilde{\mathcal{W}}(\mathcal{O}_1), \widetilde{\mathcal{W}}(\mathcal{O}_2)] = 0$,
3. *(covariance)* $\alpha_{\tilde{x},\Lambda}(\widetilde{\mathcal{W}}(\mathcal{O})) = \widetilde{\mathcal{W}}(\Lambda\mathcal{O} + \tilde{x})$, *for all* $(\tilde{x}, \Lambda) \subset \mathcal{P}_+^{\uparrow}$,
4. *(generating property)* $\widetilde{\mathcal{W}} = \overline{\bigcup_{\mathcal{O}\subset\mathbb{R}^{d+1}} \widetilde{\mathcal{W}}(\mathcal{O})}$.

Remark 1.65 *The above properties are called the Haag-Kastler axioms. Any triple*

$$\mathbb{R}^{d+1} \supset \mathcal{O} \mapsto \mathfrak{A}(\mathcal{O}), \quad \mathfrak{A}, \quad \mathcal{P}_+^{\uparrow} \ni (\tilde{x}, \Lambda) \mapsto \alpha_{(\tilde{x},\Lambda)}, \tag{201}$$

(not necessarily coming from a Weyl algebra), satisfying the above properties is called a Haag-Kastler net of C^*-algebras.

Proof. 1. is obvious from definition of $\mathcal{W}(O)$.
2. For $\mathcal{O}_1, \mathcal{O}_2$ translated double-cones (no Lorentz transformations involved) this follows from Theorem 1.63. General case: Homework.
3. Recall definition of $\mathcal{W}(O)$:

$$\mathcal{W}(\mathcal{O}) := *{-}\mathrm{Alg}\{\, \alpha_{(\tilde{x}',\Lambda')}(\mathcal{W}(O_r))\, |(\tilde{x}', \Lambda')O_r \subset \mathcal{O}\}. \tag{202}$$

Then we have

$$\begin{aligned} \alpha_{\tilde{x},\Lambda}(\mathcal{W}(\mathcal{O})) &= *{-}\mathrm{Alg}\{\, \alpha_{(\tilde{x},\Lambda)(\tilde{x}',\Lambda')}(\mathcal{W}(O_r))\, |(\tilde{x}', \Lambda')O_r \subset \mathcal{O}\} \\ &= *{-}\mathrm{Alg}\{\, \alpha_{(\tilde{x},\Lambda)(\tilde{x}',\Lambda')}(\mathcal{W}(O_r))\, |(\tilde{x}, \Lambda)(\tilde{x}', \Lambda')O_r \subset (\tilde{x}, \Lambda)\mathcal{O}\} \\ &= *{-}\mathrm{Alg}\{\, \alpha_{(\tilde{x}'',\Lambda'')}(\mathcal{W}(O_r))\, |(\tilde{x}'', \Lambda'')O_r \subset (\tilde{x}, \Lambda)\mathcal{O}\}. \end{aligned} \tag{203}$$

4. Since $\mathcal{D} = C_0^\infty(\mathbb{R}^d)$, every Weyl operator $W(f) \in \mathcal{W}$ belongs to $\widetilde{\mathcal{W}}(\mathcal{O})$ for sufficiently large $\mathcal{O}$. Thus we have $\mathcal{W} \subset \bigcup_{\mathcal{O}\subset\mathbb{R}^{d+1}} \mathcal{W}(\mathcal{O})$ and inclusions survive taking closures. The opposite inclusion is obvious. $\Box$

Theorem 1.66 *The (irreducible) representation ρ_{μ_m} of $\widetilde{\mathcal{W}}$ satisfies:*

1. *The automorphisms $\mathcal{P}_+^{\uparrow} \ni (\tilde{x}, \Lambda) \mapsto \alpha_{(\tilde{x},\Lambda)}$ are unitarily implemented by a (strongly continuous) group of unitaries $\mathcal{P}_+^{\uparrow} \ni (\tilde{x}, \Lambda) \mapsto U(\tilde{x}, \Lambda)$.*
2. *(positivity of energy) The joint spectrum of the generators (H, P) of $U(\tilde{x}) = U(\tilde{x}, I)$ is contained in $\overline{V}_+$.*
3. *(uniqueness of the vacuum vector) There is one (up to phase) unit vector Ω s.t. $U(\tilde{x}, \Lambda)\Omega = \Omega$.*

Remark 1.67 *A representation of an abstract Haag-Kastler net satisfying the above properties is called a vacuum representation.*

Remark 1.68 *Note that $\rho_{\mu=1}$ is not a vacuum representation because the automorphisms $\alpha := \alpha^{(\mu_m)}$ appearing in 1., which are compatible with locality, are not unitarily implemented in this representation. Automorphisms $\alpha^{\mu=1}$, constructed in Subsection 1.2.4, are unitarily implemented in $\rho_{\mu=1}$ but are not compatible with locality.*

1.3 Haag-Kastler net of von Neumann algebras

Given the Haag-Kastler net $\mathcal{O} \mapsto \widetilde{\mathcal{W}}(\mathcal{O}), \widetilde{\mathcal{W}}, \alpha$ as above, in the vacuum representation ρ_{μ_m}, one can proceed to a net of von Neumann algebras:

$$\mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) := \rho_{\mu_m}(\widetilde{\mathcal{W}}(\mathcal{O}))'', \quad \mathcal{A} := \overline{\bigcup_{\mathcal{O}\subset\mathbb{R}^{d+1}} \mathcal{A}(\mathcal{O})}, \quad \alpha_{(\tilde{x},\Lambda)} := U(\tilde{x},\Lambda) \,\cdot\, U(\tilde{x},\Lambda)^* \tag{204}$$

Note that $\mathcal{A}$ is defined only as a C^*-algebra by taking the norm closure (and not the strong closure) of the local von Neumann algebras. Strong closure would be too large: Since ρ_{μ_m} is irreducible, we have $\mathcal{A}'' = B(\mathcal{H})$ (where $\mathcal{H} = \Gamma(\mathfrak{h})$ in this case).

Theorem 1.69 *All local algebras $\mathcal{A}(\mathcal{O})$, for different open bounded $\mathcal{O}$, are $*$-isomorphic to a unique von Neumann algebra called "type III_1 hyperfinite factor".*

We are not going to prove this theorem, but let us (partially) explain the vocabulary:

1. A center $\mathcal{Z}$ of a von Neumann algebra $\mathcal{R}$ is defined as $\mathcal{Z} = \mathcal{R} \cap \mathcal{R}'$.

2. A von Neumann algebra is called a factor if its center is trivial i.e. $\mathcal{Z} = \mathbb{C}I$.

3. A von Neumann algebra is called hyperfinite if it is a weak closure of an increasing sequence of finite dimensional algebras (W^*-inductive limit of finite-dimensional algebras).

4. Classification of factors (Murray-von Neumann):

   - Def: Two projections $P_1, P_2 \in \mathcal{R}$ are called equivalent (denoted $P_1 \sim P_2$) if there exists a partial isometry $V \in \mathcal{R}$ from $\mathcal{H}_1 = P_1\mathcal{H}$ to $\mathcal{H}_2 = P_2\mathcal{H}$ s.t.

   $$P_1 = V^*V, \quad P_2 = VV^*. \tag{205}$$

   - Def: We say that $P_2 < P_1$ if the two projections are not equivalent, but there exists a subspace $\mathcal{H}_{1,1} \subset \mathcal{H}_1$ whose projection $P_{1,1}$ is in $\mathcal{R}$ and is equivalent to P_2.

- Thm: Let $\mathcal{R}$ be a factor and $P_1, P_2 \in \mathcal{R}$ projections. Then precisely one of the following holds:

$$P_1 > P_2, \quad P_1 \sim P_2, \quad P_1 < P_2. \tag{206}$$

- Thm: There exists a unique, up to normalization, dimension function $\mathrm{Dim}(\,\cdot\,)$ from projections in $\mathcal{R}$ to non-negative real numbers, s.t.
  (a) $\mathrm{Dim}P_1 = \mathrm{Dim}P_2$ if $P_1 \sim P_2$,
  (b) $\mathrm{Dim}P_1 > \mathrm{Dim}P_2$ if $P_1 > P_2$,
  (c) $\mathrm{Dim}P_1 < \mathrm{Dim}P_2$ if $P_1 < P_2$,
  (d) If $P_1P_2 = 0$ then $\mathrm{Dim}(P_1 + P_2) = \mathrm{Dim}P_1 + \mathrm{Dim}P_2$,
  (e) $\mathrm{Dim}0 = 0$.
- Def: $\mathcal{R}$ is **type** I if (suitably normalized) Dim ranges through $0, 1, 2, \ldots, n$, possibly $n = \infty$. In this case one can construct a decomposition $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ s.t. $\mathcal{R} = B(\mathcal{H}_1) \otimes 1_{\mathcal{H}_2}$.
- Def: $\mathcal{R}$ is **type** II_1 if (suitably normalized) Dim ranges through $[0, 1]$.
- Def: $\mathcal{R}$ is **type** II_∞ if Dim ranges through $[0, \infty]$.
- Def: $\mathcal{R}$ is **type** III if Dim takes only values 0 and ∞. Then all proper projections in $\mathcal{R}$ have infinite dimension and (for separable $\mathcal{H}$) are all equivalent.

Sub-index 1 in III_1 comes from a finer classification of Connes which will not be explained here. (See e.g. Chapter V of [9]).

There is a variant of Theorem 1.69 for general Haag-Kastler nets of von Neumann algebras, not necessarily coming from the Weyl algebra and free fields. Here we give an imprecise formulation:

Theorem 1.70 *Let $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ be a Haag-Kastler net of von Neumann algebras coming from a quantum field theory which has an ultraviolet fixed point and good thermal properties. Then, for any open bounded $\mathcal{O}$*

$$\mathfrak{A}(\mathcal{O}) \simeq \mathcal{R} \otimes \mathcal{Z}, \tag{207}$$

where $\mathcal{R}$ is the unique hyperfinite type III_1 factor and $\mathcal{Z}$ is the center of $\mathfrak{A}(\mathcal{O})$.

See [10] for a precise statement.

1.3.1 Interacting Weyl systems with infinitely many degrees of freedom (Outline)

1. Consider the representation ρ_{μ_m} of $\mathcal{W}$. In Subsection 1.2.5 we constructed a group of automorphisms α_t^0, implementing time-translations. (We add an index zero in this subsection to distinguish it from interacting dynamics to be defined below). That is

$$\rho_{\mu_m}(\alpha_t^0(W)) = U_0(t)\rho_{\mu_m}(W)U_0(t)^{-1}, \quad W \in \widetilde{\mathcal{W}}. \tag{208}$$

These time translations acted locally, that is if $W \in \widetilde{\mathcal{W}}(\mathcal{O}_r)$ then $\alpha_t(W) \in \widetilde{\mathcal{W}}(\mathcal{O}_{r+|t|})$.

2. We also had $U_0(t) = e^{itH_0}$, where

$$\begin{aligned} H_0 = \mathrm{d}\Gamma(\mu_m(p)) &= \int d^d k\, \mu_m(k) a^*(k) a(k) \\ &= \frac{1}{2}\int d^d x \big(: \pi^2(x) : + : \nabla\varphi^2(x) : + m^2 : \varphi^2(x) : \big), \end{aligned} \tag{209}$$

and $:(\cdots):$ means Wick ordering (shifting creation operators to the left and annihilation operators to the right, ignoring the commutators). For example

$$: (a^*(k_1)a^*(k_2) + a^*(k_1)a(k_2) + a(k_1)a^*(k_2) + a(k_1)a(k_2)) : \tag{210}$$

$$= a^*(k_1)a^*(k_2) + a^*(k_1)a(k_2) + a^*(k_2)a(k_1) + a(k_1)a(k_2). \tag{211}$$

φ, π will denote $\varphi_{\rho_{\mu_m}}, \pi_{\rho_{\mu_m}}$ in this subsection, that is:

$$\varphi(x) = \frac{1}{(2\pi)^{d/2}} \int \frac{d^d k}{\sqrt{2\mu_m(k)}} \big(e^{-ikx} a^*(k) + e^{ikx} a(k)\big), \tag{212}$$

$$\pi(x) = \frac{i}{(2\pi)^{d/2}} \int d^d k \sqrt{\frac{\mu_m(k)}{2}} \big(e^{-ikx} a^*(k) - e^{ikx} a(k)\big). \tag{213}$$

3. We would like to construct a group of automorphism governed by the (formal) Hamiltonian:

$$H = H_0 + H_I, \quad H_I := \lambda \int_{\mathbb{R}^d} d^d x : \varphi(x)^4 : \tag{214}$$

H_I is a well defined quadratic form on $D \times D$, where

$$D = \{ \Psi \in \Gamma_{\mathrm{fin}}(\mathfrak{h}) \,|\, \Psi^{(n)} \in S(\mathbb{R}^{nd}) \text{ for all } n \}. \tag{215}$$

However, it does not come from a densely defined operator containing Ω in its domain. Two problems when computing $H_I\Omega$:

- Integration over whole space generates expressions involving $\delta(k_1 + \cdots + k_4)$, which are thus not in L^2. ('Infrared (IR) problem').
- Decay of $(\mu_m(k_1)\ldots\mu_m(k_4))^{-1}$ is too slow to get a vector in L^2 for $d > 1$ ('Ultraviolet (UV) problem').

4. Solution of the UV problem: set $d = 1$.
Solution of the IR problem: Consider a family of Hamiltonians

$$H(g) = H_0 + H_I(g), \quad H_I(g) := \lambda \int_{\mathbb{R}} dx\, g(x) : \varphi(x)^4 :, \quad g \in C_0^\infty(\mathbb{R}^d)_{\mathbb{R}} \tag{216}$$

Thm: $H(g)$ are well-defined symmetric operators on D. Domains of essential self-adjointness are known. Each $H(g)$ has a unique (up to phase, normalized) ground state $\Omega_g \notin \mathbb{C}\Omega$. Ω_g tends weakly to zero when $g \to 1$.

5. We construct the dynamics as follows: For $A \in \mathcal{A}(\mathcal{O}_r)$ we set

$$\alpha_t(A) := e^{itH(g)} A e^{-itH(g)} \tag{217}$$

Thm: $\alpha_t(A)$ is independent of g provided that $g = 1$ on $\overline{B_{r+|t|}}$. Then also $\alpha_t(A) \in \mathcal{A}(\mathcal{O}_{r+|t|})$. Thus α_t can be extended to a group of automorphisms on $\mathcal{A}$ which respects the local structure. (PICTURE).

6. $U_g(t) := e^{itH(g)}$ does not implement α_t in the defining representation of $\mathcal{A}$ (i.e. on Fock space) because g must be modified depending on t. It turns out that α_t is not unitarily implemented in this representation (i.e. this is not a vacuum representation of the theory).

7. We want to find a representation of $\mathcal{A}$ s.t. α is unitarily implemented. Define the family of states on $\mathcal{A}$ as follows

$$\omega_g(A) := \langle \Omega_g, A\Omega_g \rangle, \quad A \in \mathcal{A}. \tag{218}$$

Although Ω_g itself tends weakly to zero as $g \to 1$, the states ω_g have weak* limit points (by Banach-Alaoglu theorem) which are non-zero simply because $\omega_g(1) = 1$.

8. Thm: There is a limit point ω of ω_g as $g \to 1$ s.t. the dynamics α_t is unitarily implemented in the corresponding GNS representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ by a group of unitaries $U_\omega(t) = e^{itH_\omega}$. The physical Hamiltonian H_ω is positive and $H_\omega \Omega_\omega = 0$.

9. By additional work one can extend $t \mapsto \alpha_t$ to $\mathcal{P}_+^\uparrow \ni (\tilde{x}, \Lambda) \mapsto \alpha_{(\tilde{x},\Lambda)}$ s.t. the resulting net $(\mathcal{A}, \alpha)$ satisfies the Haag-Kastler axioms and π_ω is its vacuum representation. $\mathrm{Sp}\,(H_\omega, P_\omega)$ looks like in the free scalar QFT, (at least for small λ). However, it can be shown that the resulting theory (called $(\varphi^4)_2$, where 2 stands for the dimension of spacetime $d+1$) is different from the free scalar QFT: its scattering matrix (to be defined later) is non-trivial. In particular there is non-trivial 2-body scattering. In this sense $(\varphi^4)_2$ is interacting.

10. The fact that the interacting time-translations α_t cannot be unitarily implemented in the defining representation of $\mathcal{A}$ (i.e. on the Fock space) can be expected on general grounds: In fact suppose the opposite and let $t \mapsto V(t)$ be the group of unitaries on $\Gamma(\mathfrak{h})$ implementing the interacting dynamics. Then $\tilde{\varphi}(t,x) := V(t)\varphi(x)V(t)^*$, $\tilde{\pi}(t,x) := V(t)\pi(x)V(t)^*$ would define a local relativistic quantum field (on Fock space). This gives a unitary equivalence between the free field φ, π and the 'interacting field' $\tilde{\varphi}, \tilde{\pi}$ at any fixed time

$$\tilde{\varphi}(t,x) := V(t)U_0(t)^* \varphi(t,x) U_0(t) V(t)^*, \tag{219}$$

$$\tilde{\pi}(t,x) := V(t)U_0(t)^* \pi(t,x) U_0(t) V(t)^*. \tag{220}$$

By a general result (Haag's theorem [22]) we have

$$\langle \Omega, \tilde{\varphi}(\tilde{x}_1) \dots \tilde{\varphi}(\tilde{x}_n)\Omega \rangle = \langle \Omega, \varphi(\tilde{x}_1) \dots \varphi(\tilde{x}_n)\Omega \rangle, \quad n = 1, 2, 3, 4. \tag{221}$$

But 4-point functions govern 2-particle scattering (via analytic continuation to Green functions and the LSZ reduction formulae). So there would be no 2-particle scattering in the theory of the 'interacting' field $\tilde{\varphi}, \tilde{\pi}$. (By slightly generalizing this discussion, one concludes that the representation π_ω above cannot be unitarily equivalent to the defining representation of $\mathcal{A}$).

(Somewhat imprecisely, Haag's theorem means that there is no interaction picture in local relativistic QFT).

11. Let us compare systems satisfying canonical commutation relations with finitely and infinitely many degrees of freedom:

    - Finitely many degrees of freedom:
        (a) By the von Neumann theorem only one (up to unitary equivalence) irreducible representation available (the Schrödinger representation) in which Q and P operators available.
        (b) Only for Hamiltonians quadratic in Q, P we could find a dynamics (group of automorphisms) on the Weyl algebra.
        (c) To study non-quadratic Hamiltonians we changed the algebra (from Weyl to resolvent algebra).
        (d) The interacting and the free dynamics on the resolvent algebra unitarily implemented in the same (Schrödinger) representation. That is, the interaction picture exists.
    - Infinitely many degrees of freedom:
        (a) Due to the breakdown of the von Neumann uniqueness theorem, there are many non-equivalent irreducible representations in which φ and π exist.
        (b) Some Hamiltonians non-quadratic in φ, π (e.g. $(\phi^4)_2$) give rise to a dynamics on the algebra $\mathcal{A}$ given by (204). Note that $\mathcal{A}$ is a bit larger than the Weyl algebra $\widetilde{\mathcal{W}}$ since we took the weak closures of local algebras. But $\widetilde{\mathcal{W}}$ is weakly dense in $\mathcal{A}$.
        (c) To treat such non-quadratic Hamiltonians we have to consider unitarily non-equivalent representation of $\mathcal{A}$ (Haag's theorem forces us to do it, breakdown of the von Neumann uniqueness theorem makes it possible).
        (d) The interacting and the free dynamics unitarily implemented in different (unitarily non-equivalent) representations. That is the interaction picture does not exists.
        (e) However, non-quadratic Hamiltonians giving rise to the dynamics on $\mathcal{A}$ which is compatible with the Haag-Kastler axioms, only available for $d = 1, 2$. The physically relevant case $d = 3$ is an open problem. For $d > 3$ there is a no-go theorem [12].

12. Thus the setting of CCR (canonical commutation relations) may be too narrow to describe interacting Haag-Kastler theories in physical spacetime. Many other approaches have been tried and are still tried (see [13] for an overview), so far without success. $(\varphi^4)_4$ is expected to be trivial due to severe UV problems ('Landau pole'), similar problems with QED. Promising candidates are non-abelian Yang-Mills theories due to their mild UV properties ('asymptotic freedom'). But there are difficulties in the IR regime ('confinement of gluons'). This question is a part of the Yang-Mills and Mass-Gap Millenium Problem.

13. Recent progress in $d \geq 1$ [15]: Take the Haag-Kastler net of v.N. algebras in the vacuum representation describing the free field theory: $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$, H_0, P_0. Let E be the joint spectral measure of H_0, P_0. Let Q be an antisymmetric matrix in $\mathbb{R}^{d+1}$ (i.e. $\tilde{p} \cdot Q\tilde{q} = -\tilde{q} \cdot Q\tilde{p}$, where $\cdot$ is Minkowski scalar product). For $A \in \mathcal{A}(\mathcal{O})$, where $\mathcal{O}$ is contained in the right wedge W (PICTURE) define

$$A_Q := \int dE(\tilde{p})\alpha^{(0)}_{Q\tilde{p}}(A). \tag{222}$$

Not well defined as it stands, but one can make sense out of it as a bounded operator, and as an observable localized in W. Similarly, for $A' \in \mathcal{A}(\mathcal{O}')$, $\mathcal{O}'$ contained in the left wedge, one defines

$$A'_{-Q} = \int dE(\tilde{p})\alpha^{(0)}_{-Q\tilde{p}}(A) \tag{223}$$

This can be interpreted as an observable localized in the left wedge W'. In fact, we have

$$[A_Q, A'_{-Q}] = 0. \tag{224}$$

Let $\mathcal{A}_Q(W)$ be the v.N. algebra generalted by all A_Q as above. We define $\mathcal{A}_Q(W') := \mathcal{A}_Q(W)'$, which is non-empty as it contains A'_{-Q}. We keep H_0, P_0 as in the free theory. This gives a wedge-local, relativistic theory which turns out to be interacting. (With two opposite wedges one can separate two particles and define 2-body scattering matrix. It is non-trivial).

The expressions A_Q, A'_{-Q} are called 'warped convolutions'. They are closely related to the concept of Rieffel deformations from non-commutative geometry.

2 Haag-Kastler theories

In this section we will consider a Haag-Kastler net of von Neumann algebras in a vacuum representation. It is given by the following objects:

1. A net of von Neumann algebras $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \subset B(\mathcal{H})$, labelled by open bounded subsets $\mathcal{O} \subset \mathbb{R}^{d+1}$.

2. The global C^*-algebra of this net $\mathcal{A} = \overline{\bigcup_{\mathcal{O}\subset\mathbb{R}^{d+1}} \mathcal{A}(\mathcal{O})}$.

3. A unitary representation $\mathcal{P}_+^\uparrow \ni (\tilde{x}, \Lambda) \mapsto U(\tilde{x}, \Lambda)$ on $\mathcal{H}$. (We will write $U(t, x) = U(\tilde{x}) := U(\tilde{x}, I)$).

4. The group of automorphisms $\alpha_{(\tilde{x},\Lambda)}(\,\cdot\,) := U(\tilde{x}, \Lambda) \cdot U(\tilde{x}, \Lambda)^*$ of $B(\mathcal{H})$.

These objects satisfy the following properties:

1. (isotony) $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$.

2. (locality) $\mathcal{O}_1 \times \mathcal{O}_2 \Rightarrow [\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0$, where X denotes spacelike separation.

3. (covariance) $\alpha_{(\tilde{x},\Lambda)}(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\Lambda\mathcal{O} + \tilde{x})$.

4. (irreducibility) $\mathcal{A}'' = B(\mathcal{H})$.

5. (spectrum condition) $\mathrm{Sp}(H, P) \subset \overline{V}_+$, where $U(t, x) = e^{iHt-iPx}$, $\tilde{x} = (t, x)$.

6. (uniqueness of the vacuum vector) There is a unique (up to phase) unit vector $\Omega \in \mathcal{H}$ s.t. $U(\tilde{x}, \Lambda)\Omega = \Omega$ for all $(\tilde{x}, \Lambda) \in \mathcal{P}_+^\uparrow$.

Remark 2.1 *We stress that in this section Ω is not necessarily a Fock space vacuum, $\mathcal{H}$ is not necessarily a Fock space and $\mathcal{A}(\mathcal{O})$ may not come from the Weyl algebra. Here we consider abstract Haag-Kastler nets and we will use only the above properties, unless stated otherwise.*

Remark 2.2 *Note that the covariance property implies that α leaves $\mathcal{A}$ invariant and thus is a group of automorphisms of this subalgebra of $B(\mathcal{H})$.*

2.1 Haag-Ruelle scattering theory

In this section we assume:

$$\mathrm{Sp}(H, P) = \{0\} \cup H_m \cup G_{2m} \tag{225}$$

as in the case of the massive free field or $(\varphi^4)_2$ at small λ. We want to construct vectors in $\mathcal{H}$ which describe an asymptotic configuration of several particles living on H_m.

For example, for two particles with energy-momenta near $\tilde{p}_1, \tilde{p}_2 \in H_m$ we pick ψ_1, ψ_2 in $E(\Delta_i)\mathcal{H}$, where Δ_i are small neighbourhoods of $\tilde{p}_i$. We want to construct vectors of the form

$$\psi_1 \times_{\mathrm{out}} \psi_2 \in \mathcal{H}, \tag{226}$$

$$\psi_1 \times_{\mathrm{in}} \psi_2 \in \mathcal{H}, \tag{227}$$

which describes two (Bosonic) particles which are independent at asymptotic times $t \to \infty$. Hence $\times_{\mathrm{out}}$ should have properties of a symmetric tensor product, but should take values in $\mathcal{H}$ not in $\mathcal{H} \otimes \mathcal{H}$.

How to do it, we know from our experience with the Fock space: we need to construct in our general framework certain creation operators $B^*_{1,t}, B^*_{2,t} \in \mathcal{A}$ (in this case time-dependent) s.t.

$$\lim_{t\to\pm\infty} B^*_{1,t}\Omega = \psi_1, \tag{228}$$

$$\lim_{t\to\pm\infty} B^*_{2,t}\Omega = \psi_2. \tag{229}$$

Then

$$\psi_1 \times_{\text{out}} \psi_2 = \lim_{t\to\infty} B^*_{1,t}B^*_{2,t}\Omega, \tag{230}$$

$$\psi_1 \times_{\text{in}} \psi_2 = \lim_{t\to-\infty} B^*_{1,t}B^*_{2,t}\Omega. \tag{231}$$

The two-body scattering matrix is a map defined by

$$S_2(\psi_1 \times_{\text{out}} \psi_2) = \psi_1 \times_{\text{in}} \psi_2 \tag{232}$$

$S_2 = I$ means that there is no two body scattering (free field). $S_2 \neq I$ means that there is scattering. This is the situation in $(\varphi^4)_2$ for example. Theories with $S_2 \neq I$ (or more generally with the full S-matrix $S \neq I$) are called interacting.

2.1.1 Energy-momentum transfer (Arveson spectrum)

To construct the creation operators mentioned in the previous section, we need to control the energy-momentum (EM) transfer of the operators (to get vectors with EM localized in small neighbourhoods of points on the mass hyperboloid).

Definition 2.3 *The energy-momentum transfer (or Arveson spectrum) of $B \in \mathcal{A}$, denoted $\mathrm{Sp}_B\alpha$, is defined as the support of the operator-valued distribution*

$$\check{B}(p^0, p) = (2\pi)^{-\frac{d+1}{2}} \int d^dx dt\, e^{-i(p^0t - px)} B(t,x), \tag{233}$$

where $B(t,x) = \alpha_{(t,x)}(B)$. Thus $\mathrm{Sp}_B\alpha$ is simply the support of the inverse Fourier transform of $(t,x) \mapsto \alpha_{(t,x)}(B)$. More precisely, we can write

$$\mathrm{Sp}_B\alpha = \overline{\bigcup_{\Psi,\Phi\in\mathcal{H}} \mathrm{supp}\, \langle\Psi, \check{B}(\cdot,\cdot)\Phi\rangle} \tag{234}$$

The following theorem justifies the name EM transfer:

Theorem 2.4 *Let $\Delta \subset \mathrm{Sp}(H,P)$ be a Borel subset. Then*

$$BE(\Delta) = E(\overline{\Delta + \mathrm{Sp}_B\alpha})BE(\Delta). \tag{235}$$

This will be called the EM transfer relation.

Before we list properties of the Arveson spectrum, we need the following fact:

Lemma 2.5 *Let $B \in \mathcal{A}$, $f \in S(\mathbb{R}^{d+1})$. Then*

$$B(f) = \int dtdx\, f(t,x)B(t,x), \tag{236}$$

defined as a weak integral (i.e. in the sense of matrix elements) is an element of $\mathcal{A}$. (Homework. It is important here that local algebras are von Neumann and we can use the bicommutant theorem).

Lemma 2.6 *Basic properties of the EM-transfer relation: (Here $B \in \mathcal{A}$)*

1. $\mathrm{Sp}_{B^*}\alpha = -\mathrm{Sp}_B\alpha$.
2. $\mathrm{Sp}_{B(t',x')}\alpha = \mathrm{Sp}_B\alpha$ *for* $(t',x') \in \mathbb{R}^{d+1}$.
3. $\mathrm{Sp}_{B(f)}\alpha \subset \mathrm{supp}\,\widehat{f}$, $f \in S(\mathbb{R}^{d+1})$.

Proof. Part 1 follows from the relation

$$\begin{aligned}(\check{B^*})(p^0,p) &= (2\pi)^{-\frac{d+1}{2}} \int d^dxdt\, e^{-i(p^0t-px)}B^*(t,x) \\ &= (2\pi)^{-\frac{d+1}{2}} \Big(\int d^dxdt\, e^{i(p^0t-px)}B(t,x) \Big)^* \\ &= \big((\check{B})(-p^0,-p)\big)^*. \end{aligned} \tag{237}$$

Part 2 follows from the change of variables

$$\begin{aligned} B(\check{t'},x')(p^0,p) &= (2\pi)^{-\frac{d+1}{2}} \int d^dxdt\, e^{-i(p^0t-px)}B(t+t',x+x') \\ &= e^{i(p^0t'-px')}(2\pi)^{-\frac{d+1}{2}} \int d^dxdt\, e^{-i(p^0(t+t')-p(x+x'))}B(t+t',x+x') \\ &= e^{i(p^0t'-px')}\check{B}(p^0,p). \end{aligned} \tag{238}$$

This distribution has clearly the same support as $\check{B}(p^0,p)$ because $e^{i(p^0t'-px')} \neq 0$.

Part 3 is a consequence of (238) and the following computation. Recall that

$$B(f) = \int d^dx'dt'\, B(t',x')f(t',x') \tag{239}$$

Hence

$$\begin{aligned} B(\check{f})(p^0,p) &= \int d^dx'dt'\, B(\check{t'},x')(p^0,p)f(t',x') \\ &= \int d^dx'dt'\, e^{i(p^0t'-px')}\check{B}(p^0,p)f(t',x') \\ &= (2\pi)^{\frac{d+1}{2}}\widehat{f}(p^0,p)\check{B}(p^0,p). \end{aligned} \tag{240}$$

□

To construct 'creation operators' of particles, one should thus pick some $A \in \mathcal{A}$ and $f \in S(\mathbb{R}^{d+1})$ s.t. supp $\widehat{f}$ is in a small neigbourhood $\Delta_{\tilde{p}}$ of some point $\tilde{p} \in H_m$ and set $B = A(f)$. Then, by the EM transfer relation, we have

$$B\Omega \in E(\Delta_{\tilde{p}})\mathcal{H}. \tag{241}$$

However, there is another constraint on creation operators. We want particles to be localized excitations. This means B should have good localization properties (not just an element of $\mathcal{A}$). A seemingly natural choice is to pick B strictly local, that is

$$B \in \bigcup_{\mathcal{O}\subset\mathbb{R}^{d+1}} \mathcal{A}(\mathcal{O}) =: \mathcal{A}_{\mathrm{loc}}. \tag{242}$$

But it turns out that strictly local B cannot have compact $\mathrm{Sp}_B\,\alpha$. We need a larger class of operators, which is still not the whole $\mathcal{A}$.

2.1.2 Almost local observables

Definition 2.7 *We say that $A \in \mathcal{A}$ is almost local if there exists a sequence $A_r \in \mathcal{A}(\mathcal{O}_r)$ s.t.*

$$\|A - A_r\| = O(r^{-\infty}). \tag{243}$$

That is for any $n \in \mathbb{N}$ there is a constant c_n s.t.

$$\|A - A_r\| \leq \frac{c_n}{r^n}, \quad r > 0. \tag{244}$$

We denote the $*$-algebra of almost local observables by $\mathcal{A}_{\mathrm{a-loc}}$.

For strictly local operators the commutator is equal to zero if we shift one of them sufficiently far in space. For almost local operators we have:

Lemma 2.8 *Let $A_1, A_2 \in \mathcal{A}_{\mathrm{a-loc}}$. Then*

$$\|[A_1, A_2(y)]\| = O(|y|^{-\infty}), \quad y \in \mathbb{R}^d. \tag{245}$$

Proof. By almost locality we find $A_i - A_{i,r} = O(r^{-\infty})$, $A_{i,r} \in \mathcal{A}(\mathcal{O}_r)$. Thus we get

$$[A_1, A_2(y)] = [A_{1,r}, A_{2,r}(y)] + O(r^{-\infty}). \tag{246}$$

Setting $r = \varepsilon|y|$ we obtain that for sufficiently small $\varepsilon > 0$ and $|y|$ sufficiently large $\mathcal{O}_r$ and $\mathcal{O}_r + y$ are spacelike separated and thus the first term o the r.h.s. above is zero. □

The following theorem gives invariance properties of $\mathcal{A}_{\mathrm{a-loc}}$:

Theorem 2.9 *Let $A \in \mathcal{A}_{\mathrm{a-loc}}$. Then*

1. $A(t,x) \in \mathcal{A}_{\mathrm{a-loc}}$ *for* $(t,x) \in \mathbb{R}^{d+1}$,

2. $A(f) \in \mathcal{A}_{\mathrm{a-loc}}$ *for* $f \in S(\mathbb{R}^{d+1})$.

Proof. As for part 1, we have

$$\|A - A_r\| = O(r^{-\infty}) \tag{247}$$

hence

$$\|A(t,x) - A_r(t,x)\| = O(r^{-\infty}). \tag{248}$$

But $A_r(t,x) \in \mathcal{A}(\mathcal{O}_r + (t,x)) \subset \mathcal{A}(\mathcal{O}_{r+|t|+|x|})$. We set $A(t,x)_{r'} = A_{r'-|t|-|x|}(t,x) \in \mathcal{A}(\mathcal{O}_{r'})$ for $r' > |t| + |x|$. For $r' \leq |t| + |x|$ we can define $A(t,x)_{r'}$ arbitrarily, e.g. $A(t,x)_{r'} = I$.

Concerning part 2, we pick $\chi_{+,r} \in C_0^\infty(\mathbb{R}^{d+1})$ s.t. $\chi_{+,r} = 1$ for $|t| + |x| < r$ and $\chi_{+,r} = 0$ for $|t| + |x| > 2r$ and set $\chi_{-,r} = 1 - \chi_{+,r}$. We write

$$f_{\pm,r}(t,x) = \chi_{\pm,r}(t,x) f(t,x), \quad f(t,x) = f_{+,r}(t,x) + f_{-,r}(t,x) \tag{249}$$

and correspondingly

$$A(f) = A(f_{+,r}) + A(f_{-,r}). \tag{250}$$

Since $f \in S(\mathbb{R}^{d+1})$ we have

$$\|A(f_{-,r})\| \leq \|A\| \|f_{-,r}\|_1 = O(r^{-\infty}). \tag{251}$$

Next, since $A = A_r + O(r^{-\infty})$

$$A(f_{+,r}) = A_r(f_{+,r}) + O(r^{-\infty})\|f\|_1. \tag{252}$$

We can set $A(f)_r = A_r(f_{+,r}) \in \mathcal{A}(\mathcal{O}_{3r})$. $\Box$

2.1.3 Regular, positive-energy solutions of the KG equation

In scattering theory we compare the interacting dynamics with the free dynamics at asymptotic times. The interacting dynamics will enter via time translates of observables $t \mapsto \alpha_t(B)$, $B \in \mathcal{A}_{\mathrm{a-loc}}$ (see the next subsection). The free dynamics will enter via regular, positive energy solutions of the Klein Gordon equation:

$$g_t(x) := (2\pi)^{-\frac{d}{2}} \int d^d p \, e^{-i\mu_m(p)t + ipx} \widehat{g}(p), \quad \widehat{g} \in C_0^\infty(\mathbb{R}^d). \tag{253}$$

The velocity support of this KG wave packet is defined as

$$V(g) := \{ \nabla \mu_m(p) \,|\, p \in \operatorname{supp} \widehat{g} \}. \tag{254}$$

Proposition 2.10 *Let $\chi_+ \in C_0^\infty(\mathbb{R}^d)$ be equal to one on $V(g)$ and zero outside of a slightly larger set and let $\chi_- = 1 - \chi_+$. We write $\chi_{\pm,t}(x) = \chi_\pm(x/t)$. Then we have*

1. $\sup_{x\in\mathbb{R}^d} |g_t(x)| = O(t^{-d/2})$,
2. $\|\chi_{-,t} g_t\|_1 = O(t^{-\infty})$,
3. $\|\chi_{+,t} g_t\|_1 = O(t^{d/2})$,
4. $\|g_t\|_1 = O(t^{d/2})$.

2.1.4 Haag-Ruelle creation operators

From now on to the end of the next subsection we follow [16].

Definition 2.11 *(a) Let $B^* \in \mathcal{A}_{\mathrm{a-loc}}$ be s.t. $\mathrm{Sp}_{B^*}\alpha \subset \overline{V_+}$ is compact, $\mathrm{Sp}_{B^*}\alpha \cap \mathrm{Sp}(H, P) \subset H_m$. Such B^* will be called a creation operator.*
(b) Let B^* be a creation operator and g_t be given by (253). Then

$$B_t^*\{g_t\} := \int d^d x\, B_t^*(x) g_t(x), \quad B_t^*(x) := U(t,x) B^* U(t,x)^*, \tag{255}$$

is called a Haag-Ruelle (HR) creation operator. We also write $B\{\overline{g}_t\} := (B^*\{g_t\})^*$.

Remark 2.12 *It is easy to see that, since $\mathrm{Sp}_{B^*}\alpha$ is compact, $t \mapsto B_t^*$ is smooth in norm and its derivatives are again creation operators. In fact, recall that $\mathrm{Sp}_{B^*}\alpha = \mathrm{supp}\, \check{B}^*(\,\cdot\,)$. Thus for any $f \in S(\mathbb{R}^{d+1})$ s.t. $\widehat{f}$ is constant on $\mathrm{Sp}_{B^*}\alpha$ (and equal to $(2\pi)^{-(d+1)/2}$) we have that $B^* = B^*(f)$. Hence $B_t^* = B^*(f_t)$, where $f_t(x^0, x) = f(x^0 - t, x)$ and differentiating $t \mapsto B_t^*$ amounts to differentiating the smooth function f.*

Lemma 2.13 *HR creation operators have the following properties:*

(a) $B_t^*\{g_t\}\Omega = B^*\{g\}\Omega$,

(b) $\partial_t(B_t^*\{g_t\}) = \dot{B}_t^*\{g_t\} + B_t^*\{\dot{g}_t\}$,

(c) $\|B_t^*\{g_t\}\| = O(t^{d/2})$, $\|B_t^*\{\chi_{+,t} g_t\}\| = O(t^{d/2})$, $\|B_t\{\chi_{-,t} g_t\}\| = O(t^{-\infty})$.

where

1. $B^*\{g\} := (B_t^*\{g_t\})|_{t=0}$, *in particular $B^*\{g\}\Omega$ is time independent.*
2. $\dot{B}^* := \partial_s B_s^*|_{s=0}$.
3. $\dot{g}_t(x) := \partial_t g_t(x) = (2\pi)^{-\frac{d}{2}} \int d^d p\, e^{-i\mu_m(p)t + ipx}(-i\mu_m(p))\widehat{g}(p)$.

Proof. (a) We compute

$$\begin{aligned} B_t^*\{g_t\}\Omega &= \int d^dx\, g_t(x)B_t^*(x)\Omega = \int d^dx\, g_t(x)U(t,x)B^*U(t,x)^*\Omega \\ &= \int d^dx\, g_t(x)U(t)U(x)B^*\Omega = (2\pi)^{d/2}e^{itH}e^{-it\mu_m(P)}\widehat{g}(P)B^*\Omega. \end{aligned} \tag{256}$$

However, $B^*\Omega \in E(H_m)\mathcal{H}$ hence $e^{itH}B^*\Omega = e^{it\mu_m(P)}B^*\Omega$ and thus

$$B_t^*\{g_t\}\Omega = (2\pi)^{d/2}\widehat{g}(P)B^*\Omega = B^*\{g\}\Omega, \tag{257}$$

where in the last step we reversed the steps with $t = 0$.

(b) Leibniz rule.

(c) We compute $\|B_t\{g_t\}\| \leq \int d^dx\, \|B_t(x)\||g_t(x)| = \|B\|\|g_t\|_1$ and similarly in the remaining cases. Now the statement follows from Proposition 2.10. □

Lemma 2.14 *Let $V(g_1)$, $V(g_2)$ be disjoint and $V(g_3)$ be arbitrary. Then*

(a) $\|[B_{1,t}^*\{g_{1,t}\}, B_{2,t}^*\{g_{2,t}\}]\| = O(t^{-\infty})$,

(b) $\|[B_{1,t}^*\{g_{1,t}\}, [B_{2,t}^*\{g_{2,t}\}, B_{3,t}^*\{g_{3,t}\}]]\| = O(t^{-\infty})$.

The above bounds also hold if some of the $B^*\{g_t\}$ are replaced with $B\{\overline{g}_t\}$.

Proof. (a) Write $g_{i,t} = \chi_{i,+}g_{i,t} + \chi_{i,-}g_{i,t}$, $i = 1, 2$, where $\chi_{i,+}$ is equal to one on $V(g_i)$ and vanishes outside of a slightly larger set and $\chi_{i,-} = 1 - \chi_{i,+}$. Then, since $\|B_t\{\chi_{-,t}g_t\}\| = O(t^{-\infty})$, we get

$$[B_{1,t}^*\{g_{1,t}\}, B_{2,t}^*\{g_{2,t}\}] = [B_{1,t}^*\{\chi_{1,+,t}g_{1,t}\}, B_{2,t}^*\{\chi_{2,+,t}g_{2,t}\}] + O(t^{-\infty}). \tag{258}$$

By almost locality of B_1, B_2 and the fact that $|g_i(t,x)| = O(1)$ uniformly in x:

$$\begin{aligned} &\|[B_{1,t}^*\{\chi_{1,+,t}g_{1,t}\}, B_{2,t}^*\{\chi_{2,+,t}g_{2,t}\}]\| \\ \leq\ & \int d^dx_1d^dx_2\, \|[B_1^*(x_1), B_2^*(x_2)]\||\chi_{1,+,t}g_{1,t}(x_1)||\chi_{2,+,t}g_{2,t}(x_2)| \\ \leq\ & \int d^dx_1d^dx_2\, |\chi_{1,+}(x_1/t)|\frac{c_n}{|x_1 - x_2|^n}|\chi_{2,+}(x_2/t)| \qquad (259) \\ \leq\ & t^{2d}\int d^dv_1d^dv_2\, |\chi_{1,+}(v_1)|\frac{c_n}{t^n|v_1 - v_2|^n}|\chi_{2,+}(v_2)|. \qquad (260) \end{aligned}$$

$\chi_{1,+}, \chi_{2,+}$ are approximate characteristic functions of $V(g_1), V(g_2)$, so they may be chosen with compact, disjoint supports. Hence the last expression is $O(t^{-\infty})$.

(b) Decompose $\widehat{g}_3(p) = \widehat{g}_{3,1}(p) + \widehat{g}_{3,2}(p)$, using a smooth partition of unity, s.t. $V(g_{3,1}) \cap V(g_1) = \emptyset$ and $V(g_{3,2}) \cap V(g_2) = \emptyset$. Now the statement follows from (a) and the Jacobi identity. □

Lemma 2.15 *Let $B_{1,t}^*\{g_{1,t}\}$, $B_{2,t}^*\{g_{2,t}\}$ be HR creation operators. If $\mathrm{Sp}_{B_1^*}\alpha, \mathrm{Sp}_{B_2^*}\alpha$ are contained in a sufficiently small neighbourhood of H_m then*

$$B_{1,t}\{\overline{g}_{1,t}\}B_{2,t}^*\{g_{2,t}\}\Omega = \Omega\langle\Omega, B_{1,t}\{\overline{g}_{1,t}\}B_{2,t}^*\{g_{2,t}\}\Omega\rangle. \tag{261}$$

Proof. Recall the energy-momentum transfer relation (235)

$$BE(\Delta) = E(\overline{\Delta + \mathrm{Sp}_B\alpha})BE(\Delta). \tag{262}$$

and the fact that $\mathrm{Sp}_{B^*}\alpha = -\mathrm{Sp}_B\alpha$. Since smearing with g_1, g_2 can only make the energy-momentum transfer smaller, we have

$$\begin{aligned} B_{1,t}\{\overline{g}_{1,t}\}B^*_{2,t}\{g_{2,t}\}\Omega &= B_{1,t}\{\overline{g}_{1,t}\}E(\mathrm{Sp}_{B_2^*}\alpha)B^*_{2,t}\{g_{2,t}\}\Omega \\ &= E(\mathrm{Sp}_{B_2^*}\alpha + \mathrm{Sp}_{B_1}\alpha)B_{1,t}\{\overline{g}_{1,t}\}E(\mathrm{Sp}_{B_2^*}\alpha)B^*_{2,t}\{g_{2,t}\}\Omega \\ &= E(\mathrm{Sp}_{B_2^*}\alpha - \mathrm{Sp}_{B_1^*}\alpha)B_{1,t}\{\overline{g}_{1,t}\}E(\mathrm{Sp}_{B_2^*}\alpha)B^*_{2,t}\{g_{2,t}\}\Omega. \end{aligned} \tag{263}$$

We know from Problem 1 of HS5 that $(H_m - H_m) \cap \mathrm{Sp}\,(H,P) = \{0\}$ (the difference of two vectors on a mass-shell is spacelike or zero). Since $\{0\}$ is isolated from the rest of the spectrum, we also have $(\mathrm{Sp}_{B_2^*}\alpha - \mathrm{Sp}_{B_1^*}\alpha) \cap \mathrm{Sp}\,(H,P) = \{0\}$, if $\mathrm{Sp}_{B_1^*}\alpha, \mathrm{Sp}_{B_2^*}\alpha$ are in small neighbourhoods of H_m. □

Note that this relation corresponds to $a(f)a^*(f)\Omega = \Omega\langle\Omega, a(f)a^*(f)\Omega\rangle$.

2.1.5 Scattering states and their Fock space structure

Theorem 2.16 *Let $B_1^*, \ldots, B_n^*$ be creation operators and $g_1, \ldots, g_n$ be regular positive energy KG wave packets with disjoint velocity supports. Then there exists the n-*particle scattering state*:*

$$\Psi^+ = \lim_{t\to\infty} B^*_{1,t}\{g_{1,t}\} \ldots B^*_{n,t}\{g_{n,t}\}\Omega. \tag{264}$$

The state Ψ^+ depends only on the single-particle vectors $\Psi_i = B^*_{i,t}\{g_{i,t}\}\Omega$, (which are time independent by Lemma 2.13 (a)) and possibly on the velocity supports $V(g_i)$[2]. Therefore we can write

$$\Psi^+ = \Psi_1 \times_{\mathrm{out}} \cdots \times_{\mathrm{out}} \Psi_n. \tag{265}$$

Proof. Let $\Psi_t := B^*_{1,t}\{g_{1,t}\} \ldots B^*_{n,t}\{g_{n,t}\}\Omega$. To show the existence of the limit $\lim_{t\to\infty}\Psi_t$ we use the Cook's method: Suppose we know that

$$\|\partial_t\Psi_t\| \leq \frac{C}{t^{1+\eta}}, \quad \eta > 0. \tag{266}$$

Then we can verify the Cauchy condition for convergence as follows:

$$\|\Psi_{t_1} - \Psi_{t_2}\| = \|\int_{t_1}^{t_2} dt\, \partial_t\Psi_t\| \leq \int_{t_1}^{t_2} dt\|\partial_t\Psi_t\| \leq \int_{t_1}^{t_2} dt\, \frac{C}{t^{1+\eta}} = C'\left(\frac{1}{t_1^\eta} - \frac{1}{t_2^\eta}\right). \tag{267}$$

This is arbitrarily small for $t_1, t_2 \geq T$, T sufficiently large so the Cauchy condition holds.

[2]This latter dependence will be excluded below.

Thus it suffices to check (266). We have

$$\begin{aligned}\partial_t \Psi_t &= \sum_{i=1}^{n} B^*_{1,t}\{g_{1,t}\}\dots\partial_t(B^*_{i,t}\{g_{i,t}\})\dots B^*_{n,t}\{g_{n,t}\}\Omega \\ &= \sum_{i=1}^{n}\sum_{j=i+1}^{n} B^*_{1,t}\{g_{1,t}\}\dots[\partial_t(B^*_{i,t}\{g_{i,t}\}), B^*_{j,t}\{g_{j,t}\}]\dots B^*_{n,t}\{g_{n,t}\}\Omega, \end{aligned} \tag{268}$$

since by Lemma 2.13 (a), $\partial_t(B^*_{i,t}(g_{i,t}))\Omega = 0$. Now we have that

$$\|B^*_{i,t}\{g_{i,t}\}\| = O(t^{d/2}), \tag{269}$$

$$\partial_t(B^*_t\{g_t\}) = \dot{B}^*_t\{g_t\} + B^*_t\{\dot{g}_t\}, \tag{270}$$

$$\|[B^*_{1,t}\{g_{1,t}\}, B^*_{2,t}\{g_{2,t}\}]\| = O(t^{-\infty}) \tag{271}$$

By sub-multiplicativity of the norm ($\|AB\| \leq \|A\|\|B\|$), we get

$$\begin{aligned}\|\partial_t\Psi_t\| &\leq \sum_{i=1}^{n}\sum_{j=i+1}^{n} \|B^*_{1,t}\{g_{1,t}\}\|\dots\|[\partial_t(B^*_{i,t}\{g_{i,t}\}), B^*_{j,t}\{g_{j,t}\}]\|\dots\|B^*_{n,t}\{g_{n,t}\}\| \\ &\leq \sum_{i=1}^{n}\sum_{j=i+1}^{n} O(t^{(n-2)d/2})O(t^{-\infty}) = O(t^{-\infty}). \end{aligned} \tag{272}$$

Second part of the theorem: let $\tilde{B}^*_{i,t}\{\tilde{g}_{i,t}\}$, $i = 1,\dots,n$ be HR creation operators satisfying the assumptions of the theorem and s.t. $\tilde{B}^*_{i,t}\{\tilde{g}_{i,t}\}\Omega = B^*_{i,t}\{g_{i,t}\}\Omega$ and velocity supports of $\tilde{g}_i$ are contained in the velocity supports of g_i. By iterating the relation

$$\begin{aligned}B^*_{1,t}\{g_{1,t}\}\dots B^*_{n,t}\{g_{n,t}\}\Omega &= B^*_{1,t}\{g_{1,t}\}\dots B^*_{n-1,t}\{g_{n-1,t}\}\tilde{B}^*_{n,t}\{\tilde{g}_{n,t}\}\Omega \\ &= \tilde{B}^*_{n,t}\{\tilde{g}_{n,t}\}B^*_{1,t}\{g_{1,t}\}\dots B^*_{n-1,t}\{g_{n-1,t}\}\Omega + O(t^{-\infty}), \end{aligned} \tag{273}$$

which follows again from (269) and from (245). By taking the limit $t \to \infty$, we obtain that Ψ^+ coincides with the scattering state $\tilde{\Psi}^+$ constructed using $\tilde{B}^*_{i,t}\{\tilde{g}_{i,t}\}$. □

Theorem 2.17 . *Let Ψ^+ and $\tilde{\Psi}^+$ be two scattering states with n and $\tilde{n}$ particles, respectively. Then*

$$\langle\tilde{\Psi}^+, \Psi^+\rangle = \delta_{n,\tilde{n}} \sum_{\sigma\in S_n} \langle\tilde{\Psi}_1, \Psi_{\sigma_1}\rangle\dots\langle\tilde{\Psi}_n, \Psi_{\sigma_n}\rangle, \tag{274}$$

$$U(\tilde{x},\Lambda)(\Psi_1 \times_{\mathrm{out}} \cdots \times_{\mathrm{out}} \Psi_n) = (U(\tilde{x},\Lambda)\Psi_1) \times_{\mathrm{out}} \cdots \times_{\mathrm{out}} (U(\tilde{x},\Lambda)\Psi_n), \tag{275}$$

where $(\tilde{x},\Lambda) \in \mathcal{P}^\uparrow_+$ and S_n is the set of permutations of an n-element set.

Proof. We assume here that $n = \tilde{n}$ (it is easy to check that the scalar product is zero otherwise by following the steps below). To prove (274), we set for simplicity of notation $B_i(t) := B_{i,t}\{\overline{g}_{i,t}\}$, $\tilde{B}_j(t) := \tilde{B}_{j,t}\{\overline{\tilde{g}}_{j,t}\}$. We assume that (274) holds for $n-1$ and compute

$$\begin{aligned} \langle \tilde{\Psi}_t, \Psi_t \rangle &= \langle \Omega, \tilde{B}_n(t) \dots \tilde{B}_1(t) B_1(t)^* \dots B_n(t)^* \Omega \rangle \\ &= \sum_{k=1}^{n} \langle \Omega, \tilde{B}_n(t) \dots \tilde{B}_2(t) B_1(t)^* \dots [\tilde{B}_1(t), B_k(t)^*] \dots B_n(t)^* \Omega \rangle \\ &= \sum_{k=1}^{n} \sum_{l=k+1}^{n} \langle \Omega, \tilde{B}_n(t) \dots \tilde{B}_2(t) B_1(t)^* \dots \check{k} \dots [[\tilde{B}_1(t), B_k(t)^*], B_l(t)^*] \dots B_n(t)^* \Omega \rangle \\ &\quad + \sum_{k=1}^{n} \langle \Omega, \tilde{B}_n(t) \dots \tilde{B}_2(t) B_1(t)^* \dots \check{k} \dots B_n(t)^* \tilde{B}_1(t) B_k(t)^* \Omega \rangle. \end{aligned} \tag{276}$$

The terms involving double commutators vanish by Lemma 2.14 (b) and Lemma 2.13 (d). In view of the last part of Theorem 2.16, we can assume without loss that all the HR creation operators involved in (276) satisfy the assumptions of Lemma 2.15. Then the last term on the r.h.s. of (276) factorizes as follows

$$\sum_{k=1}^{n} \langle \Omega, \tilde{B}_n(t) \dots \tilde{B}_2(t) B_1(t)^* \dots \check{k} \dots B_n(t)^* \Omega \rangle \langle \Omega, \tilde{B}_1(t) B_k(t)^* \Omega \rangle \tag{277}$$

by Lemma 2.15 (a). Now by the induction hypothesis the expression above factorizes in the limit $t \to \infty$ and gives (274).

Let us give some details of the last step above: Let S_n be the set of permutations of an n-element set and let $\sigma \in S_n$. Now consider all $\sigma \in S_n$ such that $\sigma_1 = k$. Each such permutation gives rise to a permutation $\rho \in S_{n-1}^{(k)}$ of an $n-1$ element set, which maps $(2, \dots, n)$ to $(1, \dots, \check{k}, \dots, n)$. ρ is given by

$$\rho_2 = \sigma_2, \dots, \rho_n = \sigma_n. \tag{278}$$

We have

$$\sum_{\sigma \in S_n} \langle \tilde{\Psi}_1, \Psi_{\sigma_1} \rangle \dots \langle \tilde{\Psi}_n, \Psi_{\sigma_n} \rangle = \sum_{k=1}^{n} \langle \tilde{\Psi}_1, \Psi_k \rangle \sum_{\rho \in S_{n-1}^{(k)}} \langle \tilde{\Psi}_2, \Psi_{\rho_2} \rangle \dots \langle \tilde{\Psi}_n, \Psi_{\rho_n} \rangle \tag{279}$$

Proof of (275) is left for the Homeworks. □

Remark 2.18 *It follows from this theorem, that scattering states depend only on single-particle states Ψ_i and not on velocity supports of g_i (cf. Theorem 2.16). Indeed, for two vectors Ψ^+ and $\tilde{\Psi}^+$ as in the last part of the proof of Theorem 2.16, we can now verify that $\|\Psi^+ - \tilde{\Psi}^+\|^2 = 0$ without assuming that velocitiy supports of $\tilde{g}_i$ are contained in velocity supports of g_i.*

2.1.6 Wave operators and the scattering matrix

From Theorem 2.16 it is not clear if any of the scattering states Ψ^+ are non-zero. Theorem 2.17 reduces this problem to showing that the HR creation operators create non-zero single-particle states from the vacuum. Actually they create a dense set in the single-particle subspace $\mathcal{H}_m := E(H_m)\mathcal{H}$.

Lemma 2.19 *Vectors of the form $B_t^*\{g_t\}\Omega$ form a dense subspace of the single-particle space $\mathcal{H}_m$.*

Proof. First we recall the definition of HR creation operators:

Definition 2.20 *(a) Let $B^* \in \mathcal{A}_{\mathrm{a-loc}}$ be s.t.* $\mathrm{Sp}_{B^*}\alpha \subset \overline{V_+}$ *is compact,* $\mathrm{Sp}_{B^*}\alpha \cap \mathrm{Sp}(H,P) \subset H_m$*. Such B^* will be called a creation operator.*
(b) Let B^* be a creation operator and g_t be given by (253). Then

$$B_t^*\{g_t\} := \int d^dx\, B_t^*(x)g_t(x), \quad B_t^*(x) := U(t,x)B^*U(t,x)^*, \tag{280}$$

is called a Haag-Ruelle (HR) creation operator.

Now let $\mathfrak{A}_{\mathrm{loc}} := \bigcup_{\mathcal{O}\subset\mathbb{R}^{d+1}} \mathcal{A}(\mathcal{O})$ be the $*$-algebra of strictly local operators. By definition of $\mathcal{A}$, $\mathcal{A}_{\mathrm{loc}}$ is norm dense in $\mathcal{A}$ and by irreducibility of $\mathcal{A}$ it is also irreducible, in particular Ω is cyclic for $\mathcal{A}_{\mathrm{loc}}$.

Note that for any $A \in \mathfrak{A}_{\mathrm{loc}}$ and $f \in S(\mathbb{R}^{d+1})$ s.t. $\mathrm{supp}\widehat{f}$ compact and $\mathrm{supp}\widehat{f} \cap \mathrm{Sp}(H,P) \subset H_m$ we have that $B^* = A(f)$ is a creation operator. Moreover

$$B^*\Omega = (2\pi)^{(d+1)/2}\widehat{f}(H,P)A\Omega. \tag{281}$$

Next, we know from the proof of Lemma 2.13, that

$$B_t^*\{g_t\}\Omega = (2\pi)^{d/2}\widehat{g}(P)B^*\Omega = (2\pi)^{d/2}(2\pi)^{(d+1)/2}\widehat{g}(P)\widehat{f}(H,P)A\Omega. \tag{282}$$

We can choose g_n, f_n within the above restrictions s.t. $\text{s-}\lim_{n\to\infty} g_n(P)f_n(H,P) = E(H_n)$. Since $A\Omega$ can approximate in norm arbitrary vector in $\mathcal{H}$, this proves the claim. □

By a small modification of the above argument one gets:

Lemma 2.21 *Vectors of the form $B_t^*\{g_t\}\Omega$, where* $\mathrm{Sp}_{B^*}\alpha \subset \Delta$ *and $\Delta \subset \overline{V}_+$ is a Borel set, form a dense subspace in $E(\Delta)\mathcal{H}_m$.*

Let $\Gamma(\mathcal{H}_m)$ be the symmetric Fock space over $\mathcal{H}_m$. Let $\Psi_1,\dots,\Psi_n$ be as in Theorem 2.16. We define the outgoing *wave operator* $W^{\mathrm{out}} : \Gamma(\mathcal{H}_m) \to \mathcal{H}$ by extending by linearity the relation:

$$W^{\mathrm{out}}(a^*(\Psi_1)\cdots a^*(\Psi_n)\Omega) = \Psi_1 \times_{\mathrm{out}} \cdots \times_{\mathrm{out}} \Psi_n. \tag{283}$$

This operator is densely defined by Lemma 2.21, and the fact that the subspace of vectors

$$a^*(\Psi_1)\cdots a^*(\Psi_n)\Omega \in \Gamma(\mathcal{H}_m), \tag{284}$$

where $\Psi_1, \dots \Psi_n$ 'live' on disjoint subsets of H_m is dense in $\mathcal{H}_m$. (We do not prove this latter fact here, but the argument is similar as in Problem 2 of HS5).

Analogously we define the incoming wave operator:

$$W^{\mathrm{in}}(a^*(\Psi_1)\cdots a^*(\Psi_n)\Omega) = \Psi_1 \times_{\mathrm{in}} \cdots \times_{\mathrm{in}} \Psi_n, \tag{285}$$

where $\Psi_1 \times_{\mathrm{in}} \cdots \times_{\mathrm{in}} \Psi_n$ is defined by taking the limit $t \to -\infty$ in Theorem 2.16 (the proof goes the same way). We also define the S-matrix: $S : \Gamma(\mathcal{H}_m) \to \Gamma(\mathcal{H}_m)$ by the formula

$$S = (W^{\mathrm{out}})^* W^{\mathrm{in}}. \tag{286}$$

1. If $S = I$ we say that the theory is non-interacting. Easy to see in free fields. If $S \neq I$ we say that the theory is interacting. Known in φ_2^4.

2. If $\mathrm{Ran}\, W^{\mathrm{out}} = \mathrm{Ran}\, W^{\mathrm{in}} = \mathcal{H}$ we say that the theory is asymptotically complete. (All vectors in $\mathcal{H}$ can be interpreted as configurations of particles). Easy to see in free fields. Partial results in φ_2^4. Interacting HK theories which are asymptotically complete constructed only recently in d=1 [18].

Remark 2.22 *There is a different definition of the scattering matrix which we used in (232): Let $\mathcal{H}^{\mathrm{in}} = \mathrm{Ran}\, W^-$ and $\mathcal{H}^{\mathrm{out}} = \mathrm{Ran}\, W^+$. Let $\tilde{S} : \mathcal{H}^{\mathrm{out}} \to \mathcal{H}^{\mathrm{in}}$ be defined as follows:*

$$\tilde{S}(\Psi_1 \times_{\mathrm{out}} \cdots \times_{\mathrm{out}} \Psi_n) = \Psi_1 \times_{\mathrm{in}} \cdots \times_{\mathrm{in}} \Psi_n. \tag{287}$$

While S and $\tilde{S}$ are different operators, their matrix elements coincide in the following sense:

$$\begin{aligned} &\langle a^*(\Psi_1)\cdots a^*(\Psi_n)\Omega, S a^*(\Psi_1')\cdots a^*(\Psi_n')\Omega\rangle \\ &= \langle \Psi_1 \times_{\mathrm{out}} \cdots \times_{\mathrm{out}} \Psi_n, \Psi_1' \times_{\mathrm{in}} \cdots \times_{\mathrm{in}} \Psi_n'\rangle \\ &= \langle \Psi_1 \times_{\mathrm{out}} \cdots \times_{\mathrm{out}} \Psi_n, \tilde{S}(\Psi_1' \times_{\mathrm{out}} \cdots \times_{\mathrm{out}} \Psi_n')\rangle. \end{aligned} \tag{288}$$

Somewhat imprecisely, one can say that asymptotic completeness is equivalent to unitarity of $\tilde{S}$ as an operator on $\mathcal{H}$. (To be meaningful, this statement requires $\mathcal{H}^{\mathrm{in}} = \mathcal{H}$, which is part of the definition of asymptotic completeness).

Let $U_m(\tilde{x}, \Lambda) := U(\tilde{x}, \Lambda)|_{\mathcal{H}_m}$. From (275), we get

$$W^{\mathrm{in/out}} \circ \Gamma(U_m(\tilde{x}, \Lambda)) = U(\tilde{x}, \Lambda) \circ W^{\mathrm{in/out}}. \tag{289}$$

Consequently, the S-matrix is Lorentz invariant:

$$S = \Gamma(U_m(\tilde{x}, \Lambda)) S \Gamma(U_m(\tilde{x}, \Lambda))^*. \tag{290}$$

2.2 Scattering theory of massless particles

Now we assume that $\mathrm{Sp}(H, P)$ is like in massless free quantum field theory i.e. $\mathrm{Sp}(H, P) = \overline{V_+}$, $\{0\}$ is a simple eigenvalue of (H, P) corresponding to the eigenvector Ω and the spectral subspace of the boundary of the lightcone H_0 contains vectors orthogonal to Ω. We denote

- $\mathcal{H}_0 := E(H_0)\mathcal{H}$.
- $\mathcal{H}_1 := \mathcal{H}_0 \cap \Omega^\perp$.

$\mathcal{H}_1$ is the single-particle subspace of the massless particles. The projection on $\mathcal{H}_1$ is denoted E_1.

Remark 2.23 *Such shape of the spectrum can also be expected in the vacuum representation of QED.*

2.2.1 Propagation properties of solutions of the KG and wave equation

Consider the following wave packets ($t \geq 0$)

$$f_t(x) = (2\pi)^{-d/2} \int d^d p \left(e^{i(\mu_m(p)t + px)} \widehat{f}_+(p) + e^{i(-\mu_m(p)t + px)} \widehat{f}_-(p) \right), \tag{291}$$

where

$$\widehat{f}_\pm(p) = \widehat{f}_1(p) \pm i\mu_m(p)\widehat{f}_2(p), \quad f_1, f_2 \in C_0^\infty(\mathbb{R}^d). \tag{292}$$

1. For $m > 0$ this is a solution of the KG equation with the following property: If f_1, f_2 supported in $G \subset \mathbb{R}^d$ then f_t supported in $G + tB_1$, where B_1 is the unit ball.

2. For $m = 0$ and $d > 1$ odd this is a solution of the wave equation with the following property: If f_1, f_2 supported in $G \subset \mathbb{R}^d$ then f_t supported in $G + tS_1$, where S_1 is the unit sphere. (Huyghens principle).

Below in this section f_t will be the solution of the wave equation, d is odd and $d > 1$.

2.2.2 Asymptotic fields

Some definitions:

1. $\mathcal{A}_{\mathrm{loc}} := \bigcup_{\mathcal{O} \subset \mathbb{R}^{d+1}} \mathcal{A}(\mathcal{O})$ is the $*$-algebra of strictly local operators.

2. For $A \in \mathcal{A}_{\mathrm{loc}}$ s.t. $\langle \Omega, A\Omega \rangle = 0$ and f_t a solution of the wave equation as in the previous subsection, we define

$$A_t\{f_t\} := \int d^d x A_t(x) f_t(x). \tag{293}$$

3. We define an averaging function h_T as follows: For $h \in C_0^\infty(\mathbb{R})$ s.t. $h \geq 0$ and $\int dt\, h(t) = 1$, we set

$$h_T(t) = \frac{1}{T^\varepsilon} h\big((t-T)/T^\varepsilon\big), \quad 0 < \varepsilon < 1. \tag{294}$$

Note that $\int dt\, h_T ...$ essentially amounts to the averaging $\frac{1}{T^\varepsilon}\int_T^{T+T^\varepsilon} dt....$ Now we set

$$A_T := \int dt\, h_T(t) A_t\{f_t\}. \tag{295}$$

Lemma 2.24 *We have*

$$\lim_{T\to\infty} A_T\Omega = (2\pi)^{d/2} E_1 f_-(P) A\Omega, \tag{296}$$

where E_1 is the projection on the single-particle subspace $\mathcal{H}_1$.

Proof. We compute

$$\begin{aligned} A_T\Omega &= \int dt\, h_T(t) A_t\{f_t\}\Omega \\ &= (2\pi)^{d/2} \int dt\, h_T(t)\, (e^{i(H+|P|)t}\widehat{f}_+(P) + e^{i(H-|P|)t}\widehat{f}_-(P))A\Omega \end{aligned} \tag{297}$$

By Lemma 2.25 below, which is a variant of the Ergodic Theorem, we get

$$\text{s-}\lim_{T\to\infty} \int dt\, h_T(t)\, e^{i(H+|P|)t} = |\Omega\rangle\langle\Omega|, \tag{298}$$

$$\text{s-}\lim_{T\to\infty} \int dt\, h_T(t)\, e^{i(H-|P|)t} = E(H_0). \tag{299}$$

However, recall that $\langle\Omega, A\Omega\rangle = 0$, so the term involving $e^{i(H+|P|)t}$ vanishes and $E(H_0)A\Omega = E_1 A\Omega$. $\square$

Let B be a self-adjoint operator and F its spectral measure. The conventional Mean Ergodic Theorem says that

$$\text{s-}\lim_{T\to\infty} \frac{1}{T}\int_0^T dt e^{itB} = F(\{0\}). \tag{300}$$

Note that $F(\{0\})$ is the subspace of all Ψ s.t. $B\Psi = 0$ or, equivalently, of all invariant vectors of $t \mapsto e^{itB}$.

Lemma 2.25 *Let B be a self-adjoint operator and F its spectral measure. Then*

$$s\text{-}\lim_{T\to\infty} \int dt\, h_T(t) e^{itB} = F(\{0\}). \tag{301}$$

Consequently, for $B_\pm = H \pm |P|$, we have

$$s\text{-}\lim_{T\to\infty} \int dt\, h_T(t)\, e^{iB_+t} = |\Omega\rangle\langle\Omega|, \tag{302}$$

$$s\text{-}\lim_{T\to\infty} \int dt\, h_T(t)\, e^{iB_-t} = E(H_0). \tag{303}$$

Proof. First, let $\Psi \in \mathrm{Ran}F(\{0\})$. Then $e^{itB}\Psi = \Psi$ and we have

$$\begin{aligned}\int dt\, h_T(t)e^{itB}\Psi &= \int dt\, h_T(t)\Psi = \int dt\, \frac{1}{T^\varepsilon}h\big((t-T)/T^\varepsilon\big)\Psi \\ &= \int dt\, \frac{1}{T^\varepsilon}h\big(t/T^\varepsilon\big)\Psi = \int dt\, h(t)\Psi = \Psi. \end{aligned} \tag{304}$$

Now let $\Phi \in \big(\mathrm{Ran}F(\{0\})\big)^\perp$. We have

$$\Phi_T := \int dt\, h_T(t)e^{itB}\Phi = (2\pi)^{1/2}\widehat{h_T}(B)\Phi = (2\pi)^{1/2}e^{iTB}\widehat{h}(T^\varepsilon B)\Phi, \tag{305}$$

where we made use of the fact that $\widehat{h_T}(\omega) = e^{iT\omega}\widehat{h}(T^\varepsilon\omega)$ (easy computation). Thus we get

$$\|\Phi_T\|^2 = (2\pi)\langle\Phi, |\widehat{h}|^2(T^\varepsilon B)\Phi\rangle \underset{T\to\infty}{\to} (2\pi)|\widehat{h}|^2(0)\langle\Phi, F(\{0\})\Phi\rangle = 0, \tag{306}$$

where we made use of properties of the spectral calculus and dominated convergence. This concludes the proof of (301).

To show the last part of the lemma, let $F_\pm$ be the spectral measure of $B_\pm$. Suppose $\Psi_+ \in \mathrm{Ran}F_+\{0\}$. Then $(H+|P|)\Psi_+ = 0$, hence $\langle\Psi_+, (H+|P|)\Psi_+\rangle = 0$, hence, since H and $|P|$ are positive

$$\langle\Psi_+, H\Psi_+\rangle = 0, \text{ and } \langle\Psi_+, |P|\Psi_+\rangle = 0. \tag{307}$$

From the first identity we get $\|H^{1/2}\Psi_+\| = 0$, therefore $H^{1/2}\Psi_+ = 0$ and consequently $H\Psi_+ = 0$. The second identity gives analogously $|P|^{1/2}\Psi_+ = 0$, hence $|P|^2\Psi_+ = 0$, hence

$$\langle\Psi_+, (P_1^2 + \cdots + P_d^2)\Psi_+\rangle = 0. \tag{308}$$

Since all terms are positive, $P_i\Psi_+ = 0$. Thus we have that Ψ_+ is an eigenvector of (H, P) with eigenvalue $\{0\}$. Since we assume that this eigenvalue is simple, we have that Ψ_+ is proportional to Ω.

Suppose $\Psi_- \in \mathrm{Ran}F_-\{0\}$. This means that $(H-|P|)\Psi_- = 0$ which just means that $\Psi_- \in \mathrm{Ran}\, E(H_0)$. $\square$

Definition 2.26 *Given an open bounded region $\mathcal{O}$ we call* future tangent *of $\mathcal{O}$ the set $V^+(\mathcal{O})$ of all points in $\mathbb{R}^{d+1}$ which have positive timelike distance from $\mathcal{O}$.*

Proposition 2.27 *Let $\mathcal{O}$ be an open bounded region and suppose that $A_{t=0}\{f_{t=0}\} \in \mathcal{A}(\mathcal{O})$ and let $B \in \mathcal{A}(\mathcal{O}_1)$, where $\mathcal{O}_1 \subset V^+(\mathcal{O})$ is an open bounded region. Then*

$$\lim_{T\to\infty} A_T B\Omega = (2\pi)^{d/2} B E_1 f_-(P) A\Omega. \tag{309}$$

Thus $A^{\mathrm{out}} := \lim_{T\to\infty} A_T$ exists on the domain

$$D(\mathcal{O}) := \{B\Omega \,|\, B \in \mathcal{A}(\mathcal{O}_1), \mathcal{O}_1 \subset V^+(\mathcal{O})\}. \tag{310}$$

as an operator.

Remark 2.28 *It turns out that $D(\mathcal{O})$ is dense.*

Proof. Let $A \in \mathcal{A}(\mathcal{O}_A)$, $\mathcal{O}_A \subset \mathbb{R}^{d+1}$, and $f_{t=0}$ be localized in a bounded region $G \subset \mathbb{R}^d$. Then

$$A_{t=0}\{f_{t=0}\} \in \mathcal{A}(\mathcal{O}_A + (0, G)), \tag{311}$$

$$A_{t=0}\{f_t\} \in \mathcal{A}(\mathcal{O}_A + (0, G + tS_1)), \tag{312}$$

$$A_t\{f_t\} \in \mathcal{A}(\mathcal{O}_A + (0, G + tS_1) + te_0) \tag{313}$$

$$= \mathcal{A}(\mathcal{O}_A + (0, G) + t(1, S_1)), \tag{314}$$

where e_0 is the unit timelike vector. We can set $\mathcal{O} = \mathcal{O}_A + (0, G)$ so that we get

$$A_t\{f_t\} \in \mathcal{A}(\mathcal{O} + t(1, S_1)). \tag{315}$$

Now the time averaging gives

$$A_T \in \bigcup_{t \in T + T^{\varepsilon} \operatorname{supp} h} \mathcal{A}(\mathcal{O} + t(1, S_1)) = \bigcup_{t \in T(1 + T^{-(1-\varepsilon)} \operatorname{supp} h)} \mathcal{A}(\mathcal{O} + t(1, S_1)). \tag{316}$$

Note that $t(1, S_1)$ are lightlike vectors. Above we take union over t in some compact interval (because supp$h$ is compact), whose length grows slower than T. Thus A_T commutes with B by locality, for T sufficiently large. By the previous lemma we have (309). □

We would like to construct scattering states using the formula

$$A_1^{\text{out}} \dots A_{n-1}^{\text{out}} A_n^{\text{out}} \Omega \tag{317}$$

but this is only possible if we have sufficient control over the domains of asymptotic fields. For example, we need to know that $A_n^{\text{out}}\Omega$ is in the domain of A_{n-1}^{out}. Furthermore, A_i^{out} are asymptotic fields and not asymptotic creation operators. One has to extract the creation parts by suitable smearing e.g. proceeding from A_i^{out} to $A_i(f)$ for $\widetilde{f}$ supported in a neighbourhood of the upper lightcone. We skip the details.

Definition 2.29 *Let $U \subset \mathbb{R}^{d+1}$ be a possibly unbounded open region and*

$$\mathcal{A}(U) := \overline{\bigcup_{\mathcal{O} \subset U} \mathcal{A}(\mathcal{O})}, \tag{318}$$

where $\mathcal{O}$ are bounded regions. We say that U has the Reeh-Schlieder property if

$$\overline{\mathcal{A}(U)\Omega} = \mathcal{H}. \tag{319}$$

Remark 2.30 *Remark 2.28 says that the future tangent $V^+(\mathcal{O})$ has the Reeh-Schlieder property. One can also show this property for other unbounded regions e.g. spacelike strings (PICTURE). One can also show the Reeh-Schlieder property for bounded regions*

$$\overline{\mathcal{A}(\mathcal{O})\Omega} = \mathcal{H} \tag{320}$$

under the additional assumption of 'weak additivity':

$$\Big(\bigcup_{\tilde{x}\in\mathbb{R}^{d+1}} \mathcal{A}(\mathcal{O}+\tilde{x}) \Big)'' = \mathcal{A}'' \textit{ for any open bounded } \mathcal{O}. \tag{321}$$

This assumption always holds if the local algebras are generated by a quantum field (as for example in the case of free fields or φ_2^4).

The Reeh-Schlieder property (320) demonstrates the non-local character of the vacuum (e.g. by measuremants here on earth one can create a state describing several particles behind the moon).

2.3 Superselection structure and statistics

We start from a Haag-Kastler net in the vacuum representation $(\mathcal{A}, \alpha)$. We denote by $\mathcal{H}_0$ the Hilbert space in the vacuum representation.

Fact: Observations cannot change the total charge of a state. If "charged particles" are created by such operations it seems to be inevitable that particles carrying opposite charge are created as well.

Consequence: The net $(\mathcal{A}, \alpha)$ on $\mathcal{H}_0$ describes states carrying the same charge as the vacuum since $\overline{\pi(\mathcal{A})\Omega} = \mathcal{H}_0$. The vacuum Ω carries no charge.

Question: Where are the states carrying a non-zero charge? Idea: these states are described by suitable representations of $(\mathcal{A}, \alpha)$. We need a criterion for selecting "interesting representations".

2.3.1 Localizable charges

Intuitively speaking, charges which have no effect at large distances, like isospin, strangeness etc. are covered. (However, electric charge is excluded). Starting point:

$$\mathcal{O} \to \mathcal{A}(\mathcal{O}) = \mathcal{A}(\mathcal{O})'' \tag{322}$$

von Neuman algebras without loss of generality. Notation: $\mathcal{A} = \overline{\bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})}^{\|\cdot\|}$.

Doplicher-Haag-Roberts [DHR] criterion: A representation $(\pi, \mathcal{H})$ of $(\mathcal{A}, \alpha)$ describes an elementary system carrying a localizable charge if:

i) $(\pi, \mathcal{H})$ is irreducible.

ii) Continuous unitary (projective) representation U of $P_+^\uparrow$ satisfying the relativistic spectral condition acts on $\mathcal{H}$:

$$U(\tilde{x}) = \int_{\overline{V_+}} e^{i\tilde{p}\tilde{x}} dE(\tilde{p}). \tag{323}$$

iii) For all double cones $\mathcal{O} \subset \mathbb{R}^{s+1}$ one has

$$\pi|_{\mathcal{A}(\mathcal{O}')} \simeq \iota|_{\mathcal{A}(\mathcal{O}')}, \tag{324}$$

where $\mathcal{O}'$ is the set of all poits spacelike separated from $\mathcal{O}$, ι is the identical (defining) representation of $\mathcal{A}$ on $\mathcal{H}_0$, i.e. $\iota(A)\Phi_0 = A\Phi_0$, $A \in \mathcal{A}$, $\Phi_0 \in \mathcal{H}_0$. Moreover,

$$\mathcal{A}(\mathcal{O}') := \overline{\bigcup_{\mathcal{O}_1 \subset \mathcal{O}'} \mathcal{A}(\mathcal{O}_1)}^{\|\cdot\|}, \tag{325}$$

where $\mathcal{O}_1$ are double cones. (Important: there is no double cone, which surrounds $\mathcal{O}$.)

Physical justification: by making measurements in $\mathcal{O}'$ one cannot determine the total charge of the state, since there may be particles carrying exactly the opposite charge which go through $\mathcal{O}$ and do not enter $\mathcal{O}'$. Phrased differently: charged states, if subject to observations (only) in $\mathcal{O}'$ cannot be distinguished from states that carry zero charge.

Questions:

* How to describe composition (”adding”) of charges?
* Why are there ”opposite” charges (antiparticles)?
* Why is there (only) Bose and Fermi statistics?
* Where are the charged (non-observable) fields?

All questions have a satisfactory answer in the case of localizable charges. (Work in progress on charges of electromagnetic type).

Definition 2.31 *A theory* $(\mathcal{A}, \alpha)$ *satisfies Haag duality if for every double cone* $\mathcal{O}$ *one has*

$$\mathcal{A}(\mathcal{O}')' = \mathcal{A}(\mathcal{O})'' = \mathcal{A}(\mathcal{O}). \tag{326}$$

Comment: $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mathcal{O}')'$ follows from locality. Haag duality may be interpreted as a maximality condition on the local algebras $\mathcal{A}(\mathcal{O})$ (i.e. one cannot add any further operators to $\mathcal{A}(\mathcal{O})$ without violating locality!). It holds in free field theories [Araki]. Later Bisognano and Wichmann established this condition in the Wightman framework of QFT.

2.3.2 DHR-Analysis

Structural analysis of all representations carrying localizable charges. First, we note that since

$$\pi|_{\mathcal{A}(\mathcal{O}')} \simeq \iota|_{\mathcal{A}(\mathcal{O}')}, \tag{327}$$

there exists an isometry $V : \mathcal{H}_0 \to \mathcal{H}$, s.t.

$$\pi(A)V = V\iota(A) = VA \text{ for } A \in \mathcal{A}(\mathcal{O}'). \tag{328}$$

Note that the unitary V depends on the choice of $\mathcal{O}$. Define a representation $(\gamma, \mathcal{H}_0)$ s.t. $\gamma(A) = V^{-1}\pi(A)V$ for all $A \in \mathcal{A}$. By construction $(\gamma, \mathcal{H}_0) \simeq (\pi, \mathcal{H})$. Moreover,

$$\gamma|_{\mathcal{A}(\mathcal{O}')} = \iota|_{\mathcal{A}(\mathcal{O}')}, \tag{329}$$

i.e. on $\mathcal{O}'$ the charged and the vacuum representations are identical. It also holds that $\gamma(\mathcal{A}(\mathcal{O}_1)) \subset \mathcal{A}(\mathcal{O}_1)$ if $\mathcal{O} \subset \mathcal{O}_1$ and hence $\gamma(\mathcal{A}) \subset \mathcal{A}$ so γ is an endomorphism of $\mathcal{A}$. In fact, suppose $A_1 \in \mathcal{A}(\mathcal{O}_1)$, $B' \in \mathcal{A}(\mathcal{O}_1')$, then

$$B'\gamma(A_1) = \gamma(B')\gamma(A_1) = \gamma(B'A_1) = \gamma(A_1B') = \gamma(A_1)\gamma(B') = \gamma(A_1)B'. \quad (330)$$

It follows that $\gamma(A_1) \in \mathcal{A}(\mathcal{O}_1')' = \mathcal{A}(\mathcal{O}_1)$, where in the last step we made use of Haag duality. We summarize:

Lemma 2.32 *Representations corresponding to localizable charges can be described by $(\gamma, \mathcal{H}_0)$, where γ is an (endo)morphism of $\mathcal{A}$. Moreover, it is localized, i.e. for some (chosen) region $\mathcal{O}$ it holds that $\gamma|_{\mathcal{A}(\mathcal{O}')} = \iota|_{\mathcal{A}(\mathcal{O}')}$.*

Intuitively, γ corresponds to 'operation' of creating a charge in $\mathcal{O}$. Let $\omega = \omega_0 \circ \gamma$, where $\omega_0(\,\cdot\,) = \langle \Omega, \cdot\, \Omega\rangle$ is the vacuum state. Then

$$\omega|_{\mathcal{A}(\mathcal{O})} \neq \omega_0, \quad (331)$$
$$\omega|_{\mathcal{A}(\mathcal{O}')} = \omega_0. \quad (332)$$

Definition 2.33 *The class of representations, which are unitarily equivalent to $(\pi, \mathcal{H})$ is called the (superselection) sector of $(\pi, \mathcal{H})$.*

Physics does not depend on a representation but only on a sector. Next, we study the relation between γ_1, γ_2 corresponding to different choices of $\mathcal{O}_1$, $\mathcal{O}_2$. By definition

$$\gamma_i(A) = V_i^{-1}\pi(A)V_i \text{ for } A \in \mathcal{A}, \quad (333)$$

where $V_i : \mathcal{H}_0 \to \mathcal{H}$. Consequently

$$V_1\gamma_1(A)V_1^{-1} = \pi(A) = V_2\gamma_2(A)V_2^{-1}, \quad (334)$$
$$V_2^{-1}V_1\gamma_1(A) = \gamma_2(A)V_2^{-1}V_1. \quad (335)$$

Note that $V_{2,1} := V_2^{-1}V_1 \in B(\mathcal{H}_0)$. For $A \in \mathcal{A}(\mathcal{O}_3')$, where $\mathcal{O}_3$ is a double cone containing $\mathcal{O}_1$ and $\mathcal{O}_2$ inside, we have $\gamma_1(A) = \gamma_2(A) = A$ i.e.

$$V_2^{-1}V_1A = AV_2^{-1}V_1. \quad (336)$$

Thus, $V_{21} = V_2^{-1}V_1 \in \mathcal{A}(\mathcal{O}_3')' = \mathcal{A}(\mathcal{O}_3)$, by the Haag duality. Conclusion: the intertwiners V_{21} are elements of the algebra of observables.

Interpretation: if ω_0 is the vacuum state, then $\omega_0\gamma_i$ describes a charged state with charge localized in $\mathcal{O}_i$:

$$\omega_0(\gamma_i(A)) = \omega_0(A) \text{ for } A \in \mathcal{A}(\mathcal{O}_i') \quad (337)$$

i.e. with regard to measurements in $\mathcal{O}_i'$ the state $\omega_0\gamma_i$ looks like the vacuum. On the other hand, $\omega_0\gamma_i$ does not belong to the vacuum sector i.e. it is charged. Phrased differently: $\omega_0\gamma_i$ describes a "charge" sitting in $\mathcal{O}_i$. Thus γ_i may be regarded as the operation of putting a charge inside of $\mathcal{O}_i$. (γ_i is "localized" in $\mathcal{O}_i$). The intertwiners are the operators of shifting the charge from $\mathcal{O}_1$ to $\mathcal{O}_2$.

2.3.3 Composite sectors

Morphisms can be composed. Let γ_1, γ_2 and $V_{21} = V_2^{-1}V_1$ as above. Then,

$$\gamma_2(\gamma_1(A)) = V_{21}\gamma_1(\gamma_1(A))V_{21}^{-1} = V_{21}\gamma_1^2(A)V_{21}^{-1} \quad (338)$$

$$\gamma_1(\gamma_2(A)) = \gamma_1(V_{21}\gamma_1(A)V_{21}^{-1}) = \gamma_1(V_{21})\gamma_1^2(A)\gamma_1(V_{21})^{-1} \quad (339)$$

Thus, $V_{21}\gamma_1(V_{21})^{-1}\gamma_1(\gamma_2(A)) = \gamma_2(\gamma_1(A))V_{21}\gamma_1(V_{21})^{-1}$ for $A \in \mathcal{A}$, hence $\gamma_1 \circ \gamma_2$ and $\gamma_2 \circ \gamma_1$ are unitarily equivalent as they are related by the intertwiners

$$\varepsilon(\gamma_1, \gamma_2) = V_{21}\gamma_1(V_{21})^{-1}. \quad (340)$$

Program: Understand the properties of $\varepsilon(\gamma_1, \gamma_2)$, which exchange the localization regions of charges.

Further simplifying assumption (not needed): $(\pi, \mathcal{H})$ describes a "simple sector" i.e. for any two equivalent representations $(\gamma_i, \mathcal{H}_{0,i})$, $i = 1, 2$ the composed representation $\gamma_1 \circ \gamma_2$ is irreducible.

Fact: $(\pi, \mathcal{H})$ describes a "simple sector" iff γ_i are automorphism of $\mathcal{A}$, i.e. are invertible.
Proof: Suppose γ_i are automorphisms. Then $\gamma_1 \circ \gamma_2$ is unitarily equivalent to the GNS representation of $\omega := \omega_0 \circ \gamma_1 \circ \gamma_2$. Recall that a GNS representation is irreducible iff the state is pure. It is easy to see that ω is pure exploiting the that that ω_0 is pure and $\gamma_1 \circ \gamma_2$ is an automorphism. Thus $(\pi, \mathcal{H})$ describes a simple sector. For the opposite implication see the computation above Theorem 2.35 below. $\square$

2.3.4 Exchange symmetry

Let γ_1, γ_2 be localized in spacelike separated regions $\mathcal{O}_1$, $\mathcal{O}_2$. We want to show that $\gamma_1 \circ \gamma_2(A) = \gamma_2 \circ \gamma_1(A)$ for $A \in \mathcal{A}$. Since $\mathcal{A} = \overline{\bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})}^{\|\cdot\|}$ it is sufficient to prove equality for $A \in \mathcal{A}(\mathcal{O})$, where $\mathcal{O}$ is arbitrarily large. Let $\mathcal{O}_3, \mathcal{O}_4 \subset \mathcal{O}'$, $\mathcal{O}_5 \subset \mathcal{O}_6'$ (such a choice of $\mathcal{O}_3 \dots \mathcal{O}_6$ is always possible. See Figure 1). Consider γ_i, $i = 1 \dots 4$, which are localized in $\mathcal{O}_i$, $i = 1 \dots 4$.

$$\gamma_1(A) = V_{13}\gamma_3(A)V_{13}^{-1} \text{ for } V_{13} \in \mathcal{A}(\mathcal{O}_5) \quad (341)$$

$$\gamma_2(A) = V_{24}\gamma_4(A)V_{24}^{-1} \text{ for } V_{24} \in \mathcal{A}(\mathcal{O}_6). \quad (342)$$

Let $A \in \mathcal{A}(\mathcal{O})$

$$\begin{aligned}\gamma_1(\gamma_2(A)) &= \gamma_1(V_{24}\gamma_4(A)V_{24}^{-1}) = V_{24}\gamma_1(A)V_{24}^{-1} = V_{24}V_{13}\gamma_3(A)V_{13}^{-1}V_{24}^{-1} \\ &= V_{13}V_{24}AV_{24}^{-1}V_{13}^{-1} = \dots = \gamma_2(\gamma_1(A)).\end{aligned} \quad (343)$$

Thus, $\gamma_1 \circ \gamma_2 = \gamma_2 \circ \gamma_1$ "operations" commute at spacelike distances.

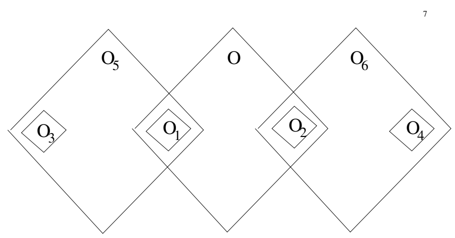

Figure 1:

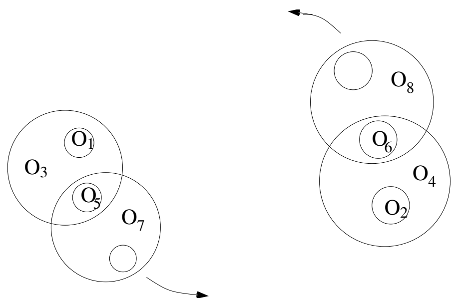

Figure 2:

2.3.5 Statistics of simple sectors

Recall that $\varepsilon(\gamma_1,\gamma_2)\gamma_1\gamma_2(A) = \gamma_2\gamma_1(A)\varepsilon(\gamma_1,\gamma_2)$ for $A \in \mathcal{A}$. Consequence: if $\mathcal{O}_1$, $\mathcal{O}_2$ are spacelike separated

$$\varepsilon(\gamma_1,\gamma_2)\gamma_1 \circ \gamma_2(A) = \gamma_1 \circ \gamma_2(A)\varepsilon(\gamma_1,\gamma_2) \text{ for } A \in \mathcal{A}. \tag{344}$$

As $(\gamma_1 \circ \gamma_2, \mathcal{H}_0)$ is by assumption irreducible, we obtain from Schur's Lemma $\varepsilon(\gamma_1,\gamma_2) \in \mathbb{T}I$, where $\mathbb{T} = \{\ z \in \mathbb{C} \mid |z| = 1\ \}$.

Consider the geometrical situation $\mathcal{O}_2 \subset \mathcal{O}_3$ and $\mathcal{O}_1 \sim \mathcal{O}_3$, (where $\sim$ denotes spacelike separation). Choose γ_3 in $\mathcal{O}_3$ which is equivalent to γ_2, i.e.

$$\gamma_3(A) = V_{32}\gamma_2(A)V_{32}^{-1} \text{ for } A \in \mathcal{A}, \tag{345}$$

where $V_{32} \in \mathcal{A}(\mathcal{O}_3)$. Then

$$\gamma_3(A) = V_{32}\gamma_2(A)V_{32}^{-1} = V_{32}V_{21}\gamma_1(A)V_{21}^{-1}V_{32}^{-1}, \tag{346}$$

so $V_{31} = V_{32}V_{21}$. It follows that

$$\begin{aligned}\varepsilon(\gamma_1,\gamma_3) &= V_{31}\gamma_1(V_{31}^{-1}) = V_{32}V_{21}\gamma_1(V_{21}^{-1}V_{32}^{-1}) \\ &= V_{21}\gamma_1(V_{21}^{-1}) = \varepsilon(\gamma_1,\gamma_2).\end{aligned} \tag{347}$$

Here we made use of the fact that $V_{21}\gamma_1(V_{21}^{-1})$ is a complex phase and that $\gamma_1(V_{32}^{-1}) = V_{32}^{-1}$.

In a similar manner, taking $\mathcal{O}_1 \subset \mathcal{O}_3$, $\mathcal{O}_3 \sim \mathcal{O}_2$, one can show that $\varepsilon(\gamma_3,\gamma_2) = \varepsilon(\gamma_1,\gamma_2)$. Thus $\varepsilon(\cdot,\cdot)$ stays constant under "local changes" of the automorphisms provided that their localization regions stay spacelike separated. Next, we note that

$$\varepsilon(\gamma_1,\gamma_2)^{-1} = (V_{21}\gamma_1(V_{21}^{-1}))^{-1} = (\gamma_2(V_{21}^{-1})V_{21})^{-1} = V_{21}^{-1}\gamma_2(V_{21}) = \varepsilon(\gamma_2,\gamma_1), \quad (348)$$

where in the second step we made use of $V_{21}\gamma_1(A) = \gamma_2(A)V_{21}$, $A \in \mathcal{A}$. Finally, we assume that the dimension of spacetime $d+1 > 2$. Deformation argument in order to exchange $\mathcal{O}_1$ and $\mathcal{O}_2$ by a sequence of double cones which stay spacelike separated: (See Figure 2 for a sequence of bases of such double cones).

$$\varepsilon(\gamma_1,\gamma_2) = \varepsilon(\gamma_3,\gamma_2) = \varepsilon(\gamma_3,\gamma_4) = \varepsilon(\gamma_5,\gamma_6) = \varepsilon(\gamma_7,\gamma_6) = \ldots = \varepsilon(\gamma_2,\gamma_1) = \varepsilon(\gamma_1,\gamma_2)^{-1}. \quad (349)$$

Summing up:

Theorem 2.34 *If the dimension of spacetime $d+1 > 2$ and $(\pi,\mathcal{H})$ is a simple sector, then either $\varepsilon(\gamma_1,\gamma_2) = 1$ when $\mathcal{O}_1 \sim \mathcal{O}_2$ or $\varepsilon(\gamma_1,\gamma_2) = -1$ when $\mathcal{O}_1 \sim \mathcal{O}_2$. (Bose-Fermi alternative in relativistic QFT).*

Remark 1. If $d+1 = 2$, analysis is possible. Strange forms of statistics appear ("braid group statistics").
Remark 2. $\gamma_1(A) = V_{12}\gamma_2(A)V_{12}^{-1} = V_{12}AV_{12}^{-1}$ if γ_2 is localized in a region spacelike separated to $\mathcal{O}$. Consequently

$$\gamma_1(A) = \lim_{"\mathcal{O}_2\to\infty"} V_{12}AV_{12}^{-1} \text{ for } A \in \bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O}). \quad (350)$$

Where "$\mathcal{O}_2 \to \infty$" means that $\mathcal{O}_2$ tends to spacelike infinity. By continuity, one has convergence for all $A \in \mathcal{A}$. (Physical picture: creation of a charge at spacelike infinity and transport of this charge by transporters V_{12} to region $\mathcal{O}_1$).
Remark 3. Composite sectors: $\gamma_1 \circ \gamma_2$ exists, providing a mechanism for "adding of charges". Let $\gamma_1, \gamma_2 \simeq \gamma$ i.e. $U_i\gamma_i(A) = \gamma(A)U_i$ for $i = 1,2$, with $U_i \in \mathcal{A}$. Then one has $\gamma_1 \circ \gamma_2 \simeq \gamma \circ \gamma = \gamma^2$. In fact

$$\begin{aligned} \gamma_1(\gamma_2(A)) &= \gamma_1(U_2\gamma(A)U_2^{-1}) = \gamma_1(U_2)\gamma_1\gamma(A)\gamma_1(U_2^{-1}) \\ &= \gamma_1(U_2)U_1\gamma(\gamma(A))U_1^{-1}\gamma_1(U_2)^{-1}. \end{aligned} \quad (351)$$

i.e. $\gamma_1(U_2)U_1$ is an intertwiner between γ^2 and $\gamma_1 \circ \gamma_2$. Thus all composite representations are equivalent. In particular $\gamma_1 \circ \gamma_2 \simeq \gamma_2 \circ \gamma_1$ with intertwiner

$$\varepsilon(\gamma_1,\gamma_2) = V_{21}\gamma_1(V_{12})^{-1}. \quad (352)$$

2.3.6 Existence of compensating ("negative") charges

Charges can be added: $\gamma_1\circ\gamma_2$, $\gamma_1\circ\gamma_2\circ\ldots\circ\gamma_n$. Structure of a semigroup. Question: can we "subtract charges"? (i.e. do we actually have a group?). Answer: yes!

One has to show that γ_1 is invertible i.e. for each $A \in \mathcal{A}(\mathcal{O})$ there exists a $B \in \mathcal{A}$ s.t. $\gamma_1(B) = A$. Chain of equalities: $A \in \mathcal{A}(\mathcal{O})$, $\mathcal{O}_1 \subset \mathcal{O}$, $\mathcal{O} \subset \mathcal{O}_2'$

$$\begin{aligned} A &= \gamma_2(A) = V_{21}\gamma_1(A)V_{21}^{-1} = \underbrace{V_{21}\gamma_1(V_{21})^{-1}}_{\varepsilon(\gamma_1,\gamma_2)=\pm1}\gamma_1(V_{21})\gamma_1(A)\gamma_1(V_{21})^{-1}\underbrace{\gamma_1(V_{21})V_{21}^{-1}}_{\varepsilon(\gamma_1,\gamma_2)^{-1}=\pm1} \\ &= \gamma_1(V_{21}AV_{21}^{-1}). \end{aligned} \quad (353)$$

Putting $B = V_{21} A V_{21}^{-1} \in \mathcal{A}$ we have $\gamma_1(B) = A$ i.e. $\gamma_1(\bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})) = \bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})$, and hence (continuity) $\gamma_1(\mathcal{A}) = \mathcal{A}$, so γ_1^{-1} exists.

Theorem 2.35 *Let $(\pi, \mathcal{H})$ describe simple localizable charges. Then the morphisms γ are automorphisms.*

Properties of γ_1^{-1}:

* As $\gamma_1(A) = A$ if $A \in \mathcal{A}(\mathcal{O}_1')$ one gets $\gamma_1^{-1}(A) = A$, $A \in \mathcal{A}(\mathcal{O}_1')$.

* Let γ_1, γ_2 be equivalent, then, denoting $V_{12} = V_{21}^{-1}$

$$\begin{aligned} V_{12}\gamma_2(A) &= \gamma_1(A)V_{12} \text{ for } A \in \mathcal{A}, \\ \gamma_2^{-1}(V_{12})A &= \gamma_2^{-1}(\gamma_1(A))\gamma_2^{-1}(V_{12}), \\ \gamma_2^{-1}(V_{12})\gamma_1^{-1}(A) &= \gamma_2^{-1}(A)\gamma_2^{-1}(V_{12}) \text{ for } A \in \mathcal{A}, \end{aligned} \tag{354}$$

  where in the last step we substituted $A \to \gamma_1^{-1}(A)$. Thus, $\gamma_2^{-1}(V_{12})$ intertwines γ_1^{-1} and γ_2^{-1}. Consequently, γ^{-1} describes a localizable charge (modulo check of covariance and spectrum condition).

* Composed sector: $(\gamma_2^{-1} \circ \gamma_1, \mathcal{H}_0)$ lies in the vacuum sector, i.e. equivalence class of $(\iota, \mathcal{H}_0)$. We have from (354) that $\gamma_2^{-1}(B) = \gamma_2^{-1}(V_{12})\gamma_1^{-1}(B)(\gamma_2^{-1}(V_{12}))^{-1}$, $B \in \mathcal{A}$. Setting $B := \gamma_1(A)$ we get

$$\gamma_2^{-1} \circ \gamma_1(A) = \gamma_2^{-1}(V_{12})A(\gamma_2^{-1}(V_{12}))^{-1} \text{ for } A \in \mathcal{A}. \tag{355}$$

* Statistics of the "opposite charge". Suppose that γ_1, γ_2 are localized in spacelike separated regions so that we have $\gamma_1 \circ \gamma_2 = \gamma_2 \circ \gamma_1$. We note, that $\gamma_1 \circ \gamma_2 = \gamma_2 \circ \gamma_1$ trivially implies $\gamma_1^{-1} \circ \gamma_2^{-1} = \gamma_2^{-1} \circ \gamma_1^{-1}$. Then, making use of $\varepsilon(\gamma_1, \gamma_2) = V_{21}\gamma_1(V_{21})^{-1}$, $V_{21}\gamma_1(A) = \gamma_2(A)V_{21}$ and (354), which says that for γ_i replaced with γ_i^{-1} the intertwiner V_{21} should be replaced with $\gamma_2^{-1}(V_{12})$, we get

$$\begin{aligned} \varepsilon(\gamma_1^{-1}, \gamma_2^{-1}) &= \gamma_2^{-1}(V_{12})\gamma_1^{-1}(\gamma_2^{-1}(V_{12}))^{-1} \\ &= \gamma_2^{-1}(V_{12}\gamma_1^{-1}(V_{12}^{-1})) = \gamma_2^{-1} \circ \gamma_1^{-1}(\gamma_1(V_{12})V_{12}^{-1}) \\ &= \gamma_2^{-1} \circ \gamma_1^{-1}(V_{12} \underbrace{V_{12}^{-1}\gamma_1(V_{12})}_{\varepsilon(\gamma_1,\gamma_2)=\pm 1} V_{12}^{-1}) = \varepsilon(\gamma_1, \gamma_2), \end{aligned} \tag{356}$$

  where in the second step we used $\gamma_1^{-1} \circ \gamma_2^{-1} = \gamma_2^{-1} \circ \gamma_1^{-1}$. Thus the charge and the compensating charge have the same statistics, i.e.

$$\varepsilon(\gamma_1^{-1}, \gamma_2^{-1}) = \varepsilon(\gamma_1, \gamma_2) \text{ for } \mathcal{O}_1 \sim \mathcal{O}_2. \tag{357}$$

2.3.7 Covariance of composite representations

By assumption, there exists a continuous unitary representation U_γ of $P_+^\uparrow$ (or its covering group $\tilde{P}_+^\uparrow$ in the case of fermionic sectors) satisfying the spectrum condition and such that

$$U_\gamma(p)\gamma(A)U_\gamma(p)^{-1} = \gamma(\alpha_p(A)) \text{ for } p \in P_+^\uparrow,\ A \in \mathcal{A}. \tag{358}$$

Question: Do there exist also such representations for composite representations γ^n (and γ^{-n})?

Definition 2.36 *We call* ${}^p\gamma(\cdot) = \alpha_p \circ \gamma \circ \alpha_p^{-1}(\cdot)$ *a transported morphism.*

Lemma 2.37 *If* γ *is localized in* $\mathcal{O}$*, then* ${}^p\gamma$ *is localized in* $p\mathcal{O}$*.*

Proof. Let $A \in \mathcal{A}((p\mathcal{O})') = \mathcal{A}(p(\mathcal{O})') = \alpha_p(\mathcal{A}(\mathcal{O}'))$ (e.g. if I translate the region then its spacelike complement translates accordingly) thus $A = \alpha_p(B)$ for some $B \in \mathcal{A}(\mathcal{O}')$. Then

$${}^p\gamma(A) = \alpha_p\gamma\alpha_p^{-1}(\alpha_p(B)) = \alpha_p \circ \gamma(B) = \alpha_p(B) = A, \tag{359}$$

where we made use of the fact that γ is localized in $\mathcal{O}$. □

Making use of $\alpha_p(A) = U(p)AU(p)^{-1}$ on $\mathcal{H}_0$ one gets

$${}^p\gamma(A) = U(p)\gamma\alpha_p^{-1}(A)U(p)^{-1} = \underbrace{U(p)U_\gamma(p)^{-1}}_{\Gamma(p)^{-1}}\gamma(A)\underbrace{U_\gamma(p)U(p)^{-1}}_{\Gamma(p)} \text{ for } A \in \mathcal{A}. \tag{360}$$

$\Gamma(p) \in \mathcal{A}(\mathcal{O}_1)$, where $\mathcal{O}_1$ contains both $\mathcal{O}$ and $p\mathcal{O}$. In fact, suppose $B \in \mathcal{A}(\mathcal{O}_1')$, then

$$\Gamma(p)B\Gamma(p)^{-1} = \Gamma(p)\gamma(B)\Gamma(p)^{-1} =^p \gamma(B) = B. \tag{361}$$

Moreover, there holds the cocycle relation for Γ

$$\begin{aligned}\Gamma(p)\alpha_p(\Gamma(p')) &= U_\gamma(p)U(p)^{-1}U(p)(U_\gamma(p')U(p')^{-1})U(p)^{-1} \\ &= U_\gamma(pp')U(pp')^{-1} = \Gamma(pp').\end{aligned} \tag{362}$$

(A similar relation holds for the interaction picture $\Gamma(t)\alpha_t(\Gamma(t')) = \Gamma(t+t')$, in the context of Dyson equation in QM).

Theorem 2.38 *Representations* U_{γ^n} *for composite sectors are defined recursively as follows*

$$U_{\gamma^n}(p) = \gamma^{n-1}(\Gamma(p))U_{\gamma^{n-1}}(p) \text{ for } p \in P_+^\uparrow, \tag{363}$$

for $n \geq 2$*.*

Proof. By induction in n:

a) First we show, by induction, that $U_{\gamma^n}(p)\gamma^n(A)U_{\gamma^n}(p)^{-1} = \gamma^n(\alpha_p(A))$.

$$\begin{aligned} U_{\gamma^n}(p)\gamma^n(A)U_{\gamma^n}(p)^{-1} &= \gamma^{n-1}(\Gamma(p))U_{\gamma^{n-1}}(p)\gamma^{n-1}(\gamma(A))U_{\gamma^{n-1}}(p)^{-1}\gamma^{n-1}(\Gamma(p)^{-1}) \\ &= \gamma^{n-1}(\Gamma(p))\gamma^{n-1}(\alpha_p(\gamma(A)))\gamma^{n-1}(\Gamma(p)^{-1}) \\ &= \gamma^{n-1}(\Gamma(p)\alpha_p(\gamma(A))\Gamma(p)^{-1}) = \gamma^{n-1}(U_\gamma(p)\gamma(A)U_\gamma(p)^{-1}) \\ &= \gamma^{n-1}(\gamma(\alpha_p(A))) = \gamma^n(\alpha_p(A)) \end{aligned} \tag{364}$$

b) Next we verify the group representation property

$$\begin{aligned} U_{\gamma^n}(p)U_{\gamma^n}(p') &= \gamma^{n-1}(\Gamma(p))U_{\gamma^{n-1}}(p)\gamma^{n-1}(\Gamma(p'))U_{\gamma^{n-1}}(p') \\ &= \gamma^{n-1}(\Gamma(p))\gamma^{n-1}(\alpha_p(\Gamma(p')))U_{\gamma^{n-1}}(p)U_{\gamma^{n-1}}(p') \\ &= \gamma^{n-1}(\Gamma(p)\alpha_p(\Gamma(p')))U_{\gamma^{n-1}}(pp') \\ &= \gamma^{n-1}(\Gamma(pp'))U_{\gamma^{n-1}}(pp') = U_{\gamma^{n-1}}(pp'). \end{aligned} \tag{365}$$

c) $p \to U_{\gamma^n}(p)$ is continuous in strong operator topology (Homework. Hints: note that $\Gamma(p)$ is localized in a fixed $\mathcal{O}_1$ for all p in a small neighbourhood of a given $p_0 \in P_+^\uparrow$.

$$\gamma^{n-1}|_{\mathcal{A}(\mathcal{O}_1)} = W \cdot W^{-1}|_{\mathcal{A}(\mathcal{O}_1)}, \tag{366}$$

for some unitary $W \in \mathcal{A}$ depending on $\mathcal{O}_1$.
d) Spectrum condition (stability of states) - literature [DHR].
e) Same results hold for "conjugate sector" (compensating charges) $(\gamma^{-m}, \mathcal{H}_0)$. (Homework: first step is to determine $U_{\gamma^{-1}}(p)$). □

Conclusion: All sectors are Poincaré covariant and stable (i.e. spectrum condition holds).

Summary: Take a local, Poincaré covariant net $\mathcal{A}$: $\mathcal{O} \to \mathcal{A}(\mathcal{O})$ on $\mathcal{H}_0$, satisfying Haag-duality, for spacetime dimension $d+1 > 2$. Consider any representation describing a simple, localizable charge in the sense of DHR. Then

* Sectors can be described by localizable automorphisms γ of $\mathcal{A}$
* Composition of sectors is possible: "γ^n" (akin to the tensor products of representations in group theory).
* existence of specific intertwiners ("permutation operators") $\varepsilon(\gamma_1, \gamma_2)$.

$$\varepsilon(\gamma_1, \gamma_2) = \pm 1 \text{ if } \mathcal{O}_1 \sim \mathcal{O}_2. \tag{367}$$

  (Bose and Fermi statistics).

* Charge conjugate sectors: γ^{-1} exists.

$$\varepsilon(\gamma_1^{-1}, \gamma_2^{-1}) = \pm 1 \text{ if } \mathcal{O}_1 \sim \mathcal{O}_2. \tag{368}$$

  (Antiparticles)

* Poincaré covariance and stability.

Remarks on extension of results:

* Non-simple sectors are completely understood in $d > 2$.

* String localized charges in $d > 3$: similar results.

$$\pi|_{\mathcal{A}(S')} \simeq \iota|_{\mathcal{A}(S')}, \tag{369}$$

  where S is a spacelike cone. Cone localization is the worst possible case in theories of massive particles [Buchholz-Fredenhagen]. (Typically appears in gauge theories).

* Localizable charges in d=2. (New types of statistics appear in low dimensions: "braid group statistics"). Anyons, plektons. [Fredenhagen, Rehren, Schroer].

* Big open problem: Superselection structure of charges of electromagnetic type.

2.3.8 Construction of charged fields

Goal: Construction of an algebra of charged Bose and Fermi fields describing the sectors of (simple) localizable charges in $d > 2$. Consider a theory of a simple additive charge i.e. γ^n is not unitarily equivalent to the identity representation. (Excludes charges of multiplicative type, like univalence, where $\gamma^2 \simeq \iota$.)

(a) Construction of physical Hilbert space containing all charges $\mathcal{H} = \mathcal{H}_0 \times \mathbb{Z}$ (direct sum). A $\Psi \in \mathcal{H}$ has a form

$$\Phi = (\ldots, \Psi_{-2}, \Psi_{-1}, \Psi_0, \Psi_1, \ldots \Psi_n, \ldots), \tag{370}$$

where $\Psi_n \in \mathcal{H}_0$, $n \in \mathbb{Z}$.

$$(\Phi, \widetilde{\Phi}) = \sum_n (\Phi_n, \widetilde{\Phi}_n)_0. \tag{371}$$

In particular $\|\Phi\|^2 = \sum_n \|\Phi_n\|_0^2 < \infty$. n is the "charge". $(0, \ldots, \Phi_n, 0, \ldots, 0)$. State with charge n.

(b) Construction of a (reducible!) representation $(\pi, \mathcal{H})$ of $\mathcal{A}$, fix localized automorphism γ

$$(\pi(A)\Phi)_n := \gamma^n(A)\Phi_n. \tag{372}$$

Note: $\pi(A)$ does not change the charge. Homework: Prove that $(\pi, \mathcal{H})$ is a representation, i.e. a $*$-homomorphism from $\mathcal{A}$ into $B(\mathcal{H})$. Prove that $(\pi, \mathcal{H})$ is reducible. Hints: prove that the projection P_n defined by $(P_m\Phi)_n = \delta_{m,n}\Phi_m$ commutes with $\pi(A)$.

(c) Construction of a continuous unitary representation U_π of a Poincaré group

$$(U_\pi(p)\Phi)_n = U_{\gamma^n}(p)\Phi_n, \quad n \in \mathbb{Z}. \tag{373}$$

Covariance:

$$\begin{aligned} & (U_\pi(p)\pi(A)U_\pi(p)^{-1}\Phi)_n = U_{\gamma^n}(p)(\pi(A)U_\pi(p)^{-1}\Phi)_n \\ = \; & U_{\gamma^n}\gamma^n(A)(U_\pi(p)^{-1}\Phi)_n = U_{\gamma^n}(p)\gamma^n(A)U_{\gamma^n}(p)^{-1}\Phi_n \\ = \; & \gamma^n(\alpha_p(A))\Phi_n = (\pi(\alpha_p(A))\Phi)_n. \end{aligned} \tag{374}$$

Thus $U_\pi(p)\pi(A)U_\pi(p)^{-1} = \pi(\alpha_p(A))$ for $A \in \mathcal{A}$.
Homework: Show that $U_\pi(p)$ is a unitary representation (i.e. $U_\pi(p)U_\pi(p') = U_\pi(pp')$) which is continuous.

Charged field operators. Goal: to introduce "decent" field operators Ψ which "shuffle" the components of the vector Φ:

$$(\Psi\Phi)_n = \Phi_{n+1}, \quad n \in \mathbb{Z}. \tag{375}$$

Remark: Ψ decreases the charge by one unit. Consider a vector $\widehat{\Phi}$ of charge n_0, i.e. $\widehat{\Phi}_n = \delta_{n,n_0}\Phi_{n_0}$.

$$(\Psi\widehat{\Phi})_n = \widehat{\Phi}_{n+1} = \delta_{n+1,n_0}\Phi_{n_0} = \delta_{n,n_0-1}\Phi_{n_0}. \tag{376}$$

i.e. Ψ indeed decreases the charge "n_0" by one unit.
Homework: Prove that the Hilbert space adjoint Ψ^* of Ψ satisfies $(\Psi^*\Phi)_n = \Phi_{n-1}$, i.e. Ψ^* increases the charge by one unit.

Ψ is unitary:

$$\|\Psi\Phi\|^2 = \sum_{n=-\infty}^{\infty} \|\Phi_{n+1}\|_0^2 = \|\Phi\|^2, \tag{377}$$

so Ψ maps $\mathcal{H}$ onto $\mathcal{H}$. Moreover, the adjoint action of Ψ implements the automorphic action of γ in $(\pi, \mathcal{H})$.

$$\begin{aligned} (\Psi\pi(A)\Psi^*\Phi)_n &= (\pi(A)\Psi^*\Phi)_{n+1} = \gamma^{n+1}(A)(\Psi^*\Phi)_{n+1} \\ &= \gamma^{n+1}(A)\Phi_n = \gamma^n(\gamma(A))\Phi_n = (\pi(\gamma(A))\Phi)_n. \end{aligned} \tag{378}$$

Thus $\Psi\pi(A)\Psi^* = (\pi(\gamma(A))\Psi)_n$ for $A \in \mathcal{A}$.

A Generalization of Haag-Ruelle theory to embedded mass hyperboloids

If the mass hyperboloid and the vacuum are embedded in the continuous spectrum, (PICTURE) the HR construction does not work as it stands. Once can however adapt it [19]. This requires the use of more complicated HR creation operators, defined as follows

$$B_T^* := \int dt\, h_T(t) B_t^*\{g_t\}. \tag{379}$$

Here $B_t^*\{g_t\}$ is defined as before and $h \geq 0$, $\int dt\, h(t) = 1$, $\widetilde{h} \in C_0^\infty(\mathbb{R})$ and

$$h_T(t) = \frac{1}{T^\varepsilon} h\left(\frac{t-T}{T^\varepsilon}\right), \quad 0 < \varepsilon < 1. \tag{380}$$

(Note that $\int dt\, h_T...$ essentially amounts to the averaging $\frac{1}{T^\varepsilon}\int_T^{T+T^\varepsilon} dt...$)

It is not difficult to show that $\mathrm{Sp}_{B_T^*}\alpha$ shrinks, as $T \to \infty$, to a subset of the embedded mass hyperboloid. Thus

$$\lim_{T\to\infty} B_T^*\Omega \in E(H_m)\mathcal{H}, \tag{381}$$

but $T \mapsto B_T^*\Omega$ is no longer time-independent, hence $\partial_T B_T^*\Omega \neq 0$. Due to this complication, to prove the existence of scattering states

$$\Psi^+ := \lim_{T\to\infty} B_{1,T}^* \dots B_{n,T}^*\Omega \tag{382}$$

(with disjoint velocity supports as before) via the Cook's method one needs to assume that for any fixed $A \in \mathcal{A}_{\mathrm{loc}}$ there is $c, \varepsilon > 0$ such that for all $\delta > 0$

$$\|E(\{(p^0, p)\,|\,p^0 \geq 0,\ |p^0 - \mu_m(p)| \leq \delta\})(1 - E(H_m))A\Omega\| \leq c\delta^\epsilon. \tag{383}$$

Given this condition, the construction can also be generalized to massless particles (where H_m is H_0 i.e. the boundary of the lightcone).

Note that property (383) trivially holds if the mass hyperboloid is isolated, thus it is consistent with the Haag-Kastler axioms. However, it does not follow form Haag-Kastler axioms (there are Haag-Kastler QFTs where it is violated). This rises a question if scattering theory (for massive or massless particles) can be developed without assuming (383). This turns out to be possible for massless particles in space of odd dimension (i.e. d odd) thanks to the Huyghens principle. The construction from [21], which avoids the Cook's method altogether, will be a topic in the later part of these lectures.

B Proof of the energy-momentum transfer relation

The energy-momentum transfer relation is an old and well known result, however proofs available in the literature (e.g. [20]) are not very accessible. Here we give an elementary argument found in [17].

Theorem B.1 *Let $A \in \mathcal{A}$ and α be the group of space-time translation automorphisms unitarily implemented by U and let E be the joint spectral measure of the energy-momentum operators. We then have the* energy-momentum transfer relation

$$AE(\Delta) = E(\overline{\Delta + \mathrm{Sp}_A\alpha})AE(\Delta) \tag{384}$$

for any Borel subset $\Delta \subset \mathbb{R}^{d+1}$.

Before giving the proof let us give a heuristic argument: Recall that $E(\Delta) = \chi_\Delta(H,P)$ where χ_Δ is a characteristic function. We set $\tilde{P} = (H,P)$ and write

$$E(\Delta) = \chi_\Delta(H,P) = (2\pi)^{-\frac{(d+1)}{2}} \int d^{d+1}\tilde{x}\, e^{i\tilde{P}\cdot\tilde{x}} \check{\chi}_\Delta(\tilde{x}). \tag{385}$$

This gives

$$\begin{aligned} AE(\Delta) &= (2\pi)^{-\frac{(d+1)}{2}} \int d^{d+1}\tilde{x}\, Ae^{i\tilde{P}\cdot\tilde{x}} \check{\chi}_\Delta(\tilde{x}) \\ &= (2\pi)^{-\frac{(d+1)}{2}} \int d^{d+1}\tilde{x}\, e^{i\tilde{P}\cdot\tilde{x}} A(-\tilde{x}) \check{\chi}_\Delta(\tilde{x}) \\ &= (2\pi)^{-\frac{(d+1)}{2}} \int dE(\tilde{p}) \int d^{d+1}\tilde{x}\, e^{i\tilde{p}\cdot\tilde{x}} A(-\tilde{x}) \check{\chi}_\Delta(\tilde{x}) \\ &= (2\pi)^{\frac{(d+1)}{2}} \int dE(\tilde{p}) \int d^{d+1}\tilde{q}\, \check{A}(\tilde{p}-\tilde{q}) \chi_\Delta(\tilde{q}) \end{aligned} \tag{386}$$

Recall that $\mathrm{Sp}_A\alpha = \mathrm{supp}\,\check{A}(\cdot)$. Since $\tilde{p}-\tilde{q} \in \mathrm{Sp}_A\alpha$ and $\tilde{q} \in \Delta$ it follows from the above that $\tilde{p} \in \mathrm{Sp}_A\alpha + \Delta$. But the above computation is not rigorous - especially in the last step where we integrate an operator valued distribution $\tilde{p} \mapsto \check{A}(\tilde{p}-\tilde{q})$ w.r.t. a spectral measure.

Proof. We can assume without loss of generality that Δ is bounded. In fact, if (384) holds for bounded Δ and we have unbounded Δ_1, we can decompose it into a disjoint union of bounded Borel sets: $\Delta_1 = \bigcup_i \Delta_i$. Then, using countable additivity of the spectral measure, we have

$$\begin{aligned} AE(\Delta_1) &= \sum_i AE(\Delta_i) = \sum_i E(\overline{\Delta_i + \mathrm{Sp}_A\alpha}) AE(\Delta_i) \\ &= E(\overline{\Delta + \mathrm{Sp}_A\alpha}) \sum_i AE(\Delta_i) = E(\overline{\Delta + \mathrm{Sp}_A\alpha}) AE(\Delta_1). \end{aligned} \tag{387}$$

Thus we assume that Δ is bounded.

Next, for any $g \in S(\mathbb{R}^{d+1})$ we note that $\widehat{g}(H,P)$ has the following properties:

1. $\widehat{g}(H,P) = E(K)\widehat{g}(H,P)$ for any Borel $K \supset \mathrm{supp}(\widehat{g})$.
2. $E(\Delta) = \widehat{g}(H,P)E(\Delta)$ for any function g such that $\widehat{g}\restriction_\Delta = 1$.

Now, let $f, g \in S(\mathbb{R}^{d+1})$. We have that

$$\begin{aligned} A(f)\widehat{g}(H,P) &= (2\pi)^{-(d+1)/2} \int d^dx dt d^dy ds\, f(t,x) g(s,y) U(t,x) A U(t-s, x-y)^* \\ &= (2\pi)^{-(d+1)/2} \int d^dz dr d^dy ds\, f(r+s, z+y) g(s,y) U(s,y) \alpha_{(r,z)}(A) \\ &= \int d^dz dr\, \widehat{h}_{(r,z)}(H,P) \alpha_{(r,z)}(A), \end{aligned} \tag{388}$$

where the function $h_{(r,z)}(s,y) = f(r+s, z+y)g(s,y)$ satisfies

$$\mathrm{supp}(\widehat{h_{(r,z)}}) \subset \mathrm{supp}(\widehat{f}) + \mathrm{supp}(\widehat{g}), \tag{389}$$

because Fourier transform of a product is a convolution. Hence

$$A(f)\widehat{g}(H,P) = E\big(\mathrm{supp}(\widehat{f}) + \mathrm{supp}(\widehat{g})\big)A(f)\widehat{g}(H,P). \tag{390}$$

To proceed we need some definitions:

1. For any set $S \subset \mathbb{R}^{d+1}$ and $\epsilon > 0$, we define $S_\epsilon := \{\tilde{p} \in \mathbb{R}^{d+1} \,|\, \mathrm{dist}(\tilde{p}, S) \le \epsilon\}$.
2. S^c is the complement of S.
3. $\varphi \in C^\infty(\mathbb{R}^{d+1})$ is function which is bounded (and all its derivatives are bounded), $\varphi \restriction_{\mathrm{Sp}_A\alpha} = 1$ and $\varphi \restriction_{(\mathrm{Sp}_A\alpha)^c_\epsilon} = 0$.

   Such function can be constructed as follows: Let $\chi_{(\mathrm{Sp}_A\alpha)_\epsilon}$ be (sharp) characteristic function of $(\mathrm{Sp}_A\alpha)_\epsilon$. Let $\eta \in C_0^\infty(\mathbb{R}^{d+1})$, $\eta \ge 0$, $\int d^{d+1}\tilde{p}\,\eta(\tilde{p}) = 1$, $\mathrm{supp}\,\eta \subset \{\,\tilde{p} \in \mathbb{R}^{d+1} \,|\, |\tilde{p}| < \epsilon/2\,\}$. We set

$$\varphi(\tilde{q}) = \int d^{d+1}\tilde{p}\,\chi_{(\mathrm{Sp}_A\alpha)_\epsilon}(\tilde{q}-\tilde{p})\eta(\tilde{p}) = \int d^{d+1}\tilde{p}\,\eta(\tilde{q}-\tilde{p})\chi_{(\mathrm{Sp}_A\alpha)_\epsilon}(\tilde{p}) \tag{391}$$

   It is easy to see that for $\tilde{q} \in \mathrm{Sp}_A\alpha$ we have $\varphi(\tilde{q}) = 1$ and for $\tilde{q} \notin (\mathrm{Sp}_A\alpha)_\epsilon$ $\varphi(\tilde{q}) = 0$. Moreover, derivatives of φ is smooth (since η is smooth) and its derivatives are bounded due to the following computation

$$|\partial_i^n\varphi(\tilde{q})| = |\int d^{d+1}\tilde{p}\,\partial_i^n\eta(\tilde{q}-\tilde{p})\chi_{(\mathrm{Sp}_A\alpha)_\epsilon}(\tilde{p})| \le \|\partial_i^n\eta\|_1 \tag{392}$$

In view of the above we get a decomposition $\widehat{f} = \widehat{f}_1 + \widehat{f}_2 = \varphi\widehat{f} + (1-\varphi)\widehat{f}$, where both f_1, f_2 are Schwartz functions (here it is important that φ has bounded derivatives), and further

$$A(f) = A(f_1) + A(f_2). \tag{393}$$

By definition of the Arveson spectrum, $A(f_2) = 0$.

Let K be a compact set such that $K \cap (\overline{\Delta + \mathrm{Sp}_A\alpha}) = \emptyset$, and consider $g \in S(\mathbb{R}^{d+1})$ such that $\widehat{g} \restriction_\Delta = 1$ and $\widehat{g} \restriction_{\Delta^c_\epsilon} = 0$. Then,

$$\begin{aligned} E(K)A(f)E(\Delta) &= E(K)A(f_1)\widehat{g}(H,P)E(\Delta) \\ &= E(K)E\big(\mathrm{supp}(\widehat{f}_1) + \mathrm{supp}(\widehat{g})\big)A(f_1)\widehat{g}(H,P)E(\Delta). \end{aligned} \tag{394}$$

Since K is disjoint from the closed set $\overline{\Delta + \mathrm{Sp}_A\alpha}$, there is ϵ small enough such that

$$\mathrm{dist}\left(K, \mathrm{supp}(\widehat{f}_1) + \mathrm{supp}(\widehat{g})\right) \ge \mathrm{dist}\left(K, (\mathrm{Sp}_A\alpha)_\epsilon + \Delta_\epsilon\right) > 0, \tag{395}$$

and hence $E(K)E\big(\mathrm{supp}(\widehat{f}_1) + \mathrm{supp}(\widehat{g})\big) = 0$. Therefore, $E(K)A(f)E(\Delta) = 0$.

Finally, let $(f_n)_{n\in\mathbb{N}}$ be a sequence converging to Dirac δ:

$$f_n(t,x) = (4\pi n^{-1})^{-\frac{1}{2}(d+1)} \mathrm{e}^{-n(t^2+|x|^2)/4}. \tag{396}$$

For any $\psi, \psi' \in \mathcal{H}$,

$$\langle \psi', A(f_n)\psi \rangle = \int d^d x dt \ \langle U(t,x)\psi', AU(t,x)\psi \rangle \, f_n(t,x) \to \langle \psi', A\psi \rangle \, , \tag{397}$$

by the strong continuity of $(t,x) \mapsto U(t,x)$, $U(0,0) = 1$ and the dominated convergence theorem. Hence, for Δ, K as above,

$$E(K)AE(\Delta) = \mathrm{w-}\lim_n E(K)A(f_n)E(\Delta) = 0. \tag{398}$$

The restriction of K being compact can be lifted by considering a countable partition of the complement of $\overline{\Delta + \mathrm{Sp}_A\alpha}$ into bounded sets, so that the statement above extends to any K such that $K \cap (\overline{\Delta + \mathrm{Sp}_A\alpha}) = \emptyset$. It follows that

$$AE(\Delta)\mathcal{H} \subset \left(E\left(\mathbb{R}^{d+1} \setminus (\overline{\Delta + \mathrm{Sp}_A\alpha}) \right) \mathcal{H} \right)^{\perp} = E\left(\overline{\Delta + \mathrm{Sp}_A\alpha} \right) \mathcal{H}, \tag{399}$$

i.e. $AE(\Delta) = E(\overline{\Delta + \mathrm{Sp}_A\alpha})AE(\Delta)$ which proves the result. □

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