AQFT: Information about the course

1. Webpage:
   https://www-m5.ma.tum.de/Allgemeines/WojciechDybalski
2. Monday 12:15-13:45 in B041 and Thursday 16:15-17:45 in B045.
3. Once every two weeks there will be a homework sheet (on the webpage). First one on 28.04.2017.
4. Once every two weeks one lecture will be an exercise class for discussion of solutions of homework sheets. First one on 08.05.2017.
5. There will be typed lecture notes (on the webpage).

What is Algebraic QFT?
How is it related to other approaches to QFT?

Outline

Lorentz group

Minkowski spacetime: $(\mathbb{R}^4, \eta)$ with $\eta := \mathrm{diag}(1, -1, -1, -1)$.

1. Lorentz group: $\mathcal{L} := \mathrm{O}(1,3) := \{\, \Lambda \in \mathrm{GL}(4, \mathbb{R}) \,|\, \Lambda \eta \Lambda^{\mathrm{T}} = \eta \,\}$
2. Proper ortochronous Lorentz group: $\mathcal{L}_+^\uparrow$ - connected component of unity in $\mathcal{L}$.

   $$\mathcal{L} = \mathcal{L}_+^\uparrow \cup T\mathcal{L}_+^\uparrow \cup P\mathcal{L}_+^\uparrow \cup TP\mathcal{L}_+^\uparrow,$$

   where $T(x^0, \vec{x}) = (-x^0, \vec{x})$ and $P(x^0, \vec{x}) = (x^0, -\vec{x})$.
3. Covering group: $\widetilde{\mathcal{L}_+^\uparrow} = \mathrm{SL}(2, \mathbb{C}) = \{\tilde{\Lambda} \in \mathrm{GL}(2, \mathbb{C}) \,|\, \det \tilde{\Lambda} = 1\}$

Poincaré group

1. Poincaré group: $\mathcal{P} := \mathbb{R}^4 \rtimes \mathcal{L}$.
2. Proper ortochronous Poincaré group: $\mathcal{P}_+^\uparrow := \mathbb{R}^4 \rtimes \mathcal{L}_+^\uparrow$.
3. Covering group: $\widetilde{\mathcal{P}}_+^\uparrow = \mathbb{R}^4 \rtimes \widetilde{\mathcal{L}}_+^\uparrow = \mathbb{R}^4 \rtimes \mathrm{SL}(2,\mathbb{C})$

Symmetries of a quantum theory

1. $\mathcal{H}$ - Hilbert space of physical states.
2. For $\Psi \in \mathcal{H}$, $\|\Psi\| = 1$ define the ray $\hat{\Psi} := \{ e^{i\phi}\Psi \mid \phi \in \mathbb{R} \}$.
3. $\hat{\mathcal{H}}$ - set of rays with the ray product $[\hat{\Phi}|\hat{\Psi}] := |\langle \Phi, \Psi \rangle|^2$.

Definition

A symmetry of a quantum system is an invertible map $\hat{S} : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ s.t. $[\hat{S}\hat{\Phi}|\hat{S}\hat{\Psi}] = [\hat{\Phi}|\hat{\Psi}]$.

Relativistic Quantum Mechanics

Theorem (Wigner 31)

For any symmetry transformation $\hat{S} : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ we can find a unitary or anti-unitary operator $S : \mathcal{H} \to \mathcal{H}$ s.t. $\hat{S}\hat{\Psi} = \widehat{S\Psi}$. S is unique up to phase.

Application:

1. $\mathcal{P}_+^\uparrow$ is a symmetry of our theory i.e., $\mathcal{P}_+^\uparrow \ni (a, \Lambda) \mapsto \hat{S}(a, \Lambda)$.

2. Thus we obtain a projective unitary representation S of $\mathcal{P}_+^\uparrow$

$$S(a_1, \Lambda_1)S(a_2, \Lambda_2) = \mathrm{e}^{i\varphi_{1,2}} S((a_1, \Lambda_1)(a_2, \Lambda_2)).$$

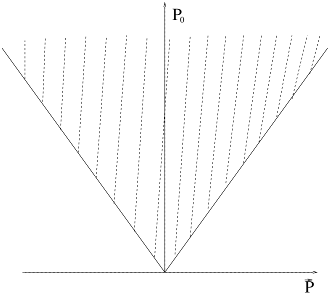
3. Fact: A projective unitary representation of $\mathcal{P}_+^\uparrow$ corresponds to an ordinary unitary representation of the covering group

$$\widetilde{\mathcal{P}}_+^\uparrow \ni (a, \tilde{\Lambda}) \mapsto U(a, \tilde{\Lambda}) \in B(\mathcal{H}).$$

Positivity of energy

Consider a unitary representation $\widetilde{\mathcal{P}}_+^\uparrow \ni (a, \tilde{\Lambda}) \mapsto U(a, \tilde{\Lambda}) \in B(\mathcal{H})$.

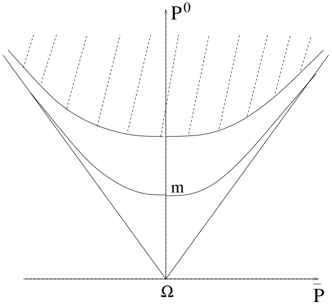
1. $P^\mu := \mathrm{i}^{-1}\partial_{a_\mu} U(a, I)|_{a=0}$ - energy momentum operators.
2. If $\mathrm{Sp}\, P \subset \overline{V}_+$ then we say that U has positive energy.



Distinguished states

1. Def: $\Omega \in \mathcal{H}$ is the vacuum state if $U(a, \tilde{\Lambda})\Omega = \Omega$ for all $(a, \tilde{\Lambda}) \in \widetilde{\mathcal{P}}_{+}^{\uparrow}$.

2. Def: $\mathcal{H}_1 \subset \mathcal{H}$ is the subspace of single-particle states of mass m and spin s if $U \restriction \mathcal{H}_1$ is the irreducible representation $[m, s]$.



Relativistic Quantum Mechanics

Definition

A relativistic quantum mechanical theory is given by:

1. $\mathcal{H}$ - Hilbert space.
2. $\widetilde{\mathcal{P}}_+^\uparrow \ni (a, \tilde{\Lambda}) \mapsto U(a, \tilde{\Lambda}) \in B(\mathcal{H})$ - a positive energy unitary rep.
3. $B(\mathcal{H})$ - possible observables.

$\mathcal{H}$ may contain a vacuum state Ω and/or subspaces of single-particle states $\mathcal{H}_{[m,s]}$.

Algebraic QFT

Definition

A relativistic (algebraic) QFT is a relativistic QM $(U, \mathcal{H})$ with a net

$$\mathbb{R}^4 \supset \mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \subset B(\mathcal{H})$$

of algebras of observables $\mathcal{A}(\mathcal{O})$ localized in open bounded regions of spacetime $\mathcal{O}$, which satisfies the Haag-Kastler postulates:

1. (Isotony) $\quad \mathcal{O}_1 \subset \mathcal{O}_2 \;\;\Rightarrow\;\; \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$,
2. (Locality) $\quad \mathcal{O}_1 \sim \mathcal{O}_2 \;\;\Rightarrow\;\; [\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = \{0\}$,
3. (Covariance) $U(a, \tilde{\Lambda})\mathcal{A}(\mathcal{O})U(a, \tilde{\Lambda})^* = \mathcal{A}(\Lambda\mathcal{O} + a)$.

Algebraic QFT

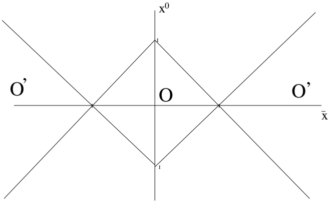
Questions:

1. Where are charges and gauge groups?
2. Where are charge-carrying (possibly anti-commuting) fields?
3. How about spin-statistics connection and CPT theorem?
4. Where are pointlike-localized fields, Green functions, path-integrals...?

Algebraic QFT

Charges and gauge groups

1. $\mathcal{A} := \overline{\bigcup_{\mathcal{O}\subset\mathbb{R}^4} \mathcal{A}(\mathcal{O})} \subset B(\mathcal{H})$.
2. Idea: Charges label 'reasonable' irreducible reps. of $\mathcal{A}$.
3. DHR criterion: π is a reasonable rep. if it looks like the vacuum representation π_0 in the spacelike complement of any 'double cone' $\mathcal{O}$.



4. 'Reasonable' reps. form a group whose dual is the global gauge group. Charge conjugation 'C':= taking inverse.

Algebraic QFT

Charge-carrying fields

Definition

A twisted-local relativistic QFT is a relativistic QM $(U, \mathcal{H}, \Omega)$

1. With algebras of charge-carrying fields $\mathcal{O} \mapsto \mathcal{F}(\mathcal{O}) \subset B(\mathcal{H})$.
2. With a unitary $k \in B(\mathcal{H})$ s.t. $k^2 = 1$ and $k\mathcal{F}(\mathcal{O})k^* \subset \mathcal{F}(\mathcal{O})$ which gives $\mathcal{F}_\pm(\mathcal{O}) := \{ F \in \mathcal{F}(\mathcal{O}) \,|\, kFk^* = \pm F \}$.

which satisfies:

1. (Isotony) $\mathcal{O}_1 \subset \mathcal{O}_2 \;\Rightarrow\; \mathcal{F}(\mathcal{O}_1) \subset \mathcal{F}(\mathcal{O}_2)$,
2. (Twisted locality) $\mathcal{O}_1 \sim \mathcal{O}_2 \;\Rightarrow\; [\mathcal{F}_\pm(\mathcal{O}_1), \mathcal{F}_\pm(\mathcal{O}_2)]_\pm = \{0\}$,
3. (Covariance) $U(a,\tilde{\Lambda})\mathcal{F}(\mathcal{O})U(a,\tilde{\Lambda})^* = \mathcal{F}(\Lambda\mathcal{O} + a)$.

Charge-carrying fields and gauge group

Theorem DHR 74, DR90

Given a Haag-Kastler QFT $(U, \mathcal{H}, \Omega, \mathcal{A})$ *one obtains:*

1. *A representation* $\pi_{\mathrm{ph}} : \mathcal{A} \to B(\mathcal{H}_{\mathrm{ph}})$ *containing 'all' DHR representations.*
2. *A twisted local relativistic QFT* $(U_{\mathrm{ph}}, \mathcal{H}_{\mathrm{ph}}, \Omega, \mathcal{F}, k)$,
3. *A compact gauge group* G *of unitary operators on* $\mathcal{H}_{\mathrm{ph}}$ *containing* k *in its center.*
4. $\pi_{\mathrm{ph}}(\mathcal{A}) = \{ F \in \mathcal{F} \mid gFg^* = F,\ g \in G \}$.

Spin-statistics connection

Theorem (Fierz 39, Pauli 40, Dell'Antonio 61...DHR 74)

1. *Suppose* $[\mathcal{F}_{+}\Omega] \supset \mathcal{H}_{\mathrm{ph},[m,s_{+}]}$. *Then* s_{+} *is integer.*
2. *Suppose* $[\mathcal{F}_{-}\Omega] \supset \mathcal{H}_{\mathrm{ph},[m,s_{-}]}$. *Then* s_{-} *is half-integer.*

CPT theorem

Theorem (Lüders 54, Pauli 55, Jost 57,...Guido-Longo 95)

Under certain additional assumptions there exists an anti-unitary operator θ on $\mathcal{H}_{\rm ph}$ which has the expected properties of the CPT operator i.e.

1. $\theta \mathcal{F}(\mathcal{O})\theta^* = \mathcal{F}(-\mathcal{O})$,
2. $\theta U_{\rm ph}(a, \tilde{\Lambda})\theta^* = U_{\rm ph}(-a, \tilde{\Lambda})$,
3. $\theta \mathcal{H}_{{\rm ph},\pi} = \mathcal{H}_{{\rm ph},\bar{\pi}}$ *and* $\theta\pi(\,\cdot\,)\theta^* = \bar{\pi}(\,\cdot\,)$,

where $\bar{\pi}$ is the charge conjugate representation of π.

Algebraic QFT

Pointlike localized fields

Definition (Fredenhagen-Hertel 81, Bostelmann 04)

A quadratic form ϕ_j is a pointlike field of a relativistic QFT, if there exists $F_{j,r} \in \mathcal{F}(\mathcal{O}_r)$, where $\mathcal{O}_r$ is the ball of radius r centered at zero, s.t.

$$\|(1+P^0)^{-\ell}(\phi_j - F_{j,r})(1+P^0)^{-\ell}\| \underset{r\to 0}{\to} 0 \quad \text{for} \quad \text{some} \quad \ell \geq 0.$$

Theorem (Bostelmann 04)

Under certain technical assumptions one obtains that

$$\phi_j(x) := U(x, I)\phi_j U(x, I)^*$$

are relativistic quantum fields in the sense of Wightman.

Wightman QFT

Definition

A Wightman QFT is a relativistic QM $(U, \mathcal{H}, \Omega)$ with distributions

$$\mathcal{S}(\mathbb{R}^4) \ni f \mapsto \phi_j(f) =: \int d^4x\, \phi_j(x) f(x) \in [\text{operators on } \mathcal{H}]$$

defining quantum fields. They satisfy

1. (Twisted locality) $\quad [\phi_j(x), \phi_k(y)]_\pm = 0$ for $x - y$ spacelike,
2. (Covariance) $\quad U(a, \tilde{\Lambda})\phi_j(x)U(a, \tilde{\Lambda})^* = D(\tilde{\Lambda}^{-1})_{j,k}\phi_k(\Lambda x + a)$,

where D is a finite-dimensional representation of $\widetilde{\mathcal{L}}_+^\uparrow$.

Wightman QFT

Irreducible representations of $\widetilde{\mathcal{L}}_+^\uparrow = SL(2, \mathbb{C})$

1. Representation space: $\mathcal{H}^{(\frac{j}{2}, \frac{k}{2})} := \mathrm{Sym}(\otimes^j \mathbb{C}^2) \otimes \mathrm{Sym}(\otimes^k \mathbb{C}^2)$
2. Representation: $D^{(\frac{j}{2}, \frac{k}{2})}(\tilde{\Lambda}) = (\otimes^j \tilde{\Lambda}) \otimes (\otimes^k \bar{\tilde{\Lambda}})$

Example 1. Some familiar fields

1. $D = D^{(0,0)}$ - scalar field φ
2. $D = D^{(\frac{1}{2}, \frac{1}{2})}$ - vector field j^μ
3. $D = D^{(\frac{1}{2}, 0)} \oplus D^{(0, \frac{1}{2})}$ - Dirac field ψ
4. $D = D^{(1,0)} \oplus D^{(0,1)}$ - Faraday tensor $F^{\mu\nu}$

Example 2. Free scalar field φ_{f}

1. Consider a scalar field which satisfies

$$(\Box + m^2)\varphi_{\mathrm{f}}(x) = 0, \quad \Box := \partial_\mu \partial^\mu .$$

2. Fact: This is the usual free scalar field on $\mathcal{H} = \Gamma(L^2(\mathbb{R}^3))$

$$\varphi_{\mathrm{f}}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\vec{p}}{\sqrt{2\omega(\vec{p})}} (\mathrm{e}^{\mathrm{i}\omega(\vec{p})x^0 - \mathrm{i}\vec{p}\vec{x}} a^*(\vec{p}) + \mathrm{e}^{-\mathrm{i}\omega(\vec{p})x^0 + \mathrm{i}\vec{p}\vec{x}} a(\vec{p})).$$

Wightman QFT

Example 3. Interacting scalar field φ

1. Consider a scalar field which satisfies

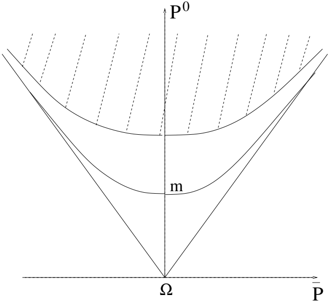
$$(\Box + m^2)\varphi(x) = -\frac{\lambda}{3!}\text{"}\varphi(x)^3\text{"}.$$

Theorem (Glimm-Jaffe 68...)

1. *This theory, called φ^4, exists in 2 and 3 dimensional spacetime and satisfies the Haag-Kastler and Wightman postulates.*
2. *Furthermore, the theory is non-trivial.*

Scattering theory

1. Consider a massive Wightman theory of a scalar field φ.



Wightman QFT

Scattering theory

1. Consider a massive Wightman theory of a scalar field φ.
2. Define a non-local field φ_ε by

$$\widetilde{\varphi}_\varepsilon(p) := \chi_{[m^2-\varepsilon, m^2+\varepsilon]}(p^2)\widetilde{\varphi}(p).$$

3. Set $a_t^*(g_t) := \int d^3\vec{x}\, \varphi_\varepsilon(t,\vec{x})\overleftrightarrow{\partial}_0 g(t,\vec{x})$ where g is a positive energy Klein-Gordon solution.

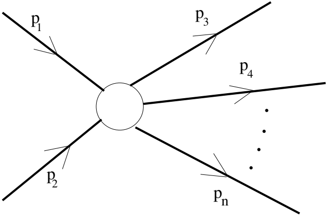
Theorem (Haag 58, Ruelle 62)

The following limits exist

$$\Psi^{\mathrm{out/in}} := \lim_{t\to +/-\infty} a_t^*(g_{1,t})\dots a_t^*(g_{n,t})\Omega$$

and span subspaces $\mathcal{H}^{\mathrm{out}}, \mathcal{H}^{\mathrm{in}} \subset \mathcal{H}$ naturally isomorphic to $\Gamma(\mathcal{H}_1)$.

Scattering matrix and Green functions (LSZ)



Theorem (Lehmann-Symanzik-Zimmermann 55, Hepp 66)

$$^{\mathrm{out}}\langle p_3, p_4, \ldots, p_n | p_1, p_2 \rangle^{\mathrm{in}} = (-i)^n \tilde{G}^{\mathrm{a,c}}(-p_1, -p_2, p_3, \ldots, p_n),$$

where $G^{\mathrm{a,c}}$ denotes connected, amputated Green functions.

Wightman QFT

Theorem (Lehmann-Symanzik-Zimmermann 55, Hepp 66)

$$^{\mathrm{out}}\langle p_3, p_4, \ldots, p_n | p_1, p_2\rangle^{\mathrm{in}} = (-i)^n \tilde{G}^{\mathrm{a,c}}(-p_1, -p_2, p_3, \ldots, p_n)$$

where $G^{\mathrm{a,c}}$ denotes connected, amputated Green functions.

Green functions

1. $G(x_1, \ldots, x_n) := \langle \Omega, \mathcal{T}\varphi(x_1)\ldots\varphi(x_n)\Omega\rangle$, where $\mathcal{T}$ is time ordering.

2. $G(x_1, \ldots, x_n) = \sum_{\pi \in P} \prod_{R\in\pi} G(x_{i_1^R}, \ldots, x_{i_{|R|}^R})^{\mathrm{c}}$, for example

$$G(x_1, x_2)^{\mathrm{c}} := G(x_1, x_2) - G(x_1)G(x_2).$$

3. $G^{\mathrm{a,c}}(x_1, \ldots, x_n) := (\Box_1 + m^2)\ldots(\Box_n + m^2)G(x_1, \ldots x_n)^{\mathrm{c}}.$

Perturbative QFT

Path-integral formula

$$G(x_1,\dots,x_n) = \text{“}\frac{1}{N}\int D\phi\,\phi(x_1)\dots\phi(x_n)\,\mathrm{e}^{iS[\phi]}\text{”},\ \ D\phi := \text{“}\prod_{x\in\mathbb{R}^4} d\phi(x)\text{”},$$

$$S[\phi] := \int d^4x\left(\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{m^2}{2}\phi(x)^2 - \frac{\lambda}{4!}\phi(x)^4\right).$$

1. Wick rotation

$$G_E(x_1,\dots,x_n) = \text{“}\frac{1}{N}\int D\phi\,\phi(x_1)\dots\phi(x_n)\,\mathrm{e}^{-S_E[\phi]},\ \text{”}$$

$$G_E(x_1,\dots,x_n) := G(-ix_1^0,\vec{x}_1,\dots,-ix_n^0,\vec{x}_n),\quad S_E[\phi]\geq 0$$

2. One wants to determine G_E as formal power series in λ:

$$G_E(x_1,\dots,x_n) = \sum_{r=0}^{\infty}\lambda^r G_{E,r}(x_1,\dots,x_n).$$

No control over convergence of the series.

Outline

Program of this course

1. Algebraic Quantum Mechanics
2. Free field theory and interacting models
3. Haag-Kastler postulates
4. Haag-Ruelle scattering theory
5. DHR theory of charged representations

Remark:
Wightman QFT and rigorous perturbative QFT will be covered in the TMP core module 'Quantum Field Theory', WS 17/18, M. Beneke/W.D.