Matrix analysis in quantum theory
Seminar SS18

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This seminar provides an introduction to basic concepts of quantum theory with an emphasis on the mathematical tools used to describe finite-dimensional systems. We will discuss fundamental information-theoretic results concerning quantum states and channels. These are derived using techniques from matrix analysis including matrix trace inequalities, functional calculus, convexity and majorization. They are associated with natural operational tasks such as communication and cryptography. We will study a few examples of quantum information processing protocols such as teleportation or dense coding.

Literature: We will mainly stick to D. Petz *Quantum Information Theory and Quantum Statistics*, Springer 2008; supplementing it with material from the reference list below.

Prerequisites: Prerequisites: Linear Algebra 2 (MA1102) Functional Analysis (MA3001) desirable, but not strictly necessary. Similarly, prior knowledge of quantum mechanics is advantageous.

List of topics and suggested references

1. **Mathematical preliminaries & their motivation in quantum mechanics:** Hilbert space, Hilbert-Schmidt scalar product, projections, self-adjoint operators and the spectral theorem, unitary and positive operators. ([1, Ch. 11.1-4.], see also: [4, Ch. I]) States and observables (basic notions), Bloch sphere ([1, Ch. 2.1, Def. (A0-3)])

2. **Tensor products & composite systems:** tensor product, partial trace, reduced density matrices ([1, Ch. 2.1], see also: [2, Ch. 5.1-4], [4, Ch. I]), Schmidt decomposition ([1, Lemma 4.1])

3. **Positive maps & quantum channels:** positive maps, completely positive maps, Kraus representation. Application: characterization of quantum channels, general state transformation, unitary quantum dynamics. ([1, Ch. 2.2, Ch. 11.7.])

4. **Convexity and monotonicity for trace functions & Gibbs states:** Monotonicity of traces, Peierls and Bogoliubov inequality, Kleins inequality. Application to von Neumann entropy: characterisation of Gibbs states. Non-negativity of the relative entropy. ([2, Ch. 1.2, 2.2-3])

5. **Operator monotonicity and convexity:** Definition and (counter-)examples. Löwner's characterization ([2, Ch 2.1.], see also: [3, Ch. 4], [4, Ch. V]), Lieb's concavity theorem. Application: joint convexity of the relative quantum entropy ([2, Ch. 6.1])

6. **Trace distance & continuity of the quantum entropy:** Trace and Schatten norms. Carlen-Lieb bound. Application: Audenaert-Fannes bound. ([1, Ch. 11.4, 3.3], [2, Ch. 7.1])

7. **Entanglement:** product, separable and entangled states, purification, Application: entanglement witness ([1, Ch. 4.1])

8. **Majorization & more mixed states:** Definition of majorization, Birkhoff's theorem ([1, Ch. 11.5], see also: [4, Ch. II.1-2]). Application: more mixed states, monotonicity of the quantum entropy for separable states ([1, Ch 4.1.])

9. **Dense coding & teleportation:** Bell basis, teleportation and dense coding protocol ([1, Ch. 4.2]), equivalence with fully depolarizing maps ([7]).

10. **Golden-Thompson inequality, Lieb's triple matrix inequality & strong subadditivity of quantum entropy:** Complete proof of strong-subadditivity via Lieb's triple matrix inequality [6]

References

[1] D. Petz, Quantum Information Theory and Quantum Statistics, Springer, 2008.

[2] E. Carlen, Trace Inequalities and Quantum Entropy: An Introductory Course, AMS Contemporary Mathematics 529, 2010.

[3] F. Hiai and D. Petz, Introduction to matrix analysis and applications, Springer, 2014

[4] R. Bhatia, Matrix analysis, Springer, 1997

[5] A. Horn and C. Johnson, Matrix analysis, Cambridge University Press, 2013

[6] M.B. Ruskai, Inequalities for Quantum Entropy: A Review with Conditions for Equality. J. Math. Phys. 43, 4358–4375 (2002); Erratum. J. Math. Phys. 46, 019901 (2005).

[7] R. F. Werner, All Teleportation and Dense Coding Schemes, J. Phys. A, vol. 34, no. 35, August 2001.