Blow-up and Long time behaviour of kinetic equations with cubic nonlinearities.

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Munich, April 1, 2017. "Macroscopic Limits of Quantum Systems". .

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Bosonic Nordheim equation (**spatially homogeneous**).

$$\partial_t F_1 = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} q(F) \mathrm{M} d^3p_2 d^3p_3 d^3p_4 \ \ , \ \ p_1 \in \mathbb{R}^3 \ \ , \ \ t > 0$$

$$F_1(0,p_1) = F_0(p_1) \ \ , \ \ p_1 \in \mathbb{R}^3$$

$$q(F) = q_3(F) + q_2(F) \quad , \quad \epsilon = \frac{|p|^2}{2}$$

$$\mathrm{M} = \mathrm{M}(p_1,p_2;p_3,p_4) = \delta(p_1 + p_2 - p_3 - p_4)\delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4)$$

$$q_3(F) = F_3F_4(F_1 + F_2) - F_1F_2(F_3 + F_4)$$

$$q_2(F) = F_3F_4 - F_1F_2$$

Notation: $F_j = F(t,p_j)$, $p_j \in \mathbb{R}^3$, $j = 1,2,3,4$.

Weak Turbulence Equation:

$$\partial_t F_1 = \int_{\mathrm{R}^3}\int_{\mathrm{R}^3}\int_{\mathrm{R}^3} q_3(F)\mathrm{M} d^3p_2 d^3p_3 d^3p_4 \ \ , \ \ p_1 \in \mathrm{R}^3 \ \ , \ \ t > 0$$

$$F_1(0,p_1) = F_0(p_1) \ \ , \ \ p \in \mathrm{R}^3$$

$$\epsilon = \frac{|p|^2}{2}$$

$$\mathrm{M} = \mathrm{M}(p_1,p_2;p_3,p_4) = \delta(p_1 + p_2 - p_3 - p_4)\delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4)$$

$$q_3(F) = F_3F_4(F_1 + F_2) - F_1F_2(F_3 + F_4)$$

Notation: $F_j = F(t,p_j)$, $p_j \in \mathrm{R}^3$, $j = 1,2,3,4$.

Nordheim equation (1928).

- Kinetic equation describing the evolution of the distribution of momentum of a dilute gas of bosons.

- Similar derivation to Boltzmann, except for that it uses Bose-Einstein statistics to compute the number of particles changing their state in the collisions.

Weak Turbulence Equation (Hasselmann, Zakharov) (1963-1965).

- Physical motivation. Description of a random field of waves which satisfy an equation with weak nonlinearities. (Example: Water waves).

$$i\partial_t \psi = -\Delta\psi + \varepsilon|\psi|^2\psi \ , \ \psi(x,0) = \text{"Random"}$$

$$\psi(x,t) = \int a(k,t)e^{ikx}dk$$

$$F(p.t) = \left\langle |a(k,t)|^2 \right\rangle$$

F satisfies the Weak Turbulence equation.

Rigorous results:

- Benedetto, Castello, Esposito, Pulvirenti.
- Lukharinen, Spohn.

Nordheim equation (**1928**).

Stationary solutions: Bose-Einstein distributions.

$$F_{BE}(p) = m_0 \delta(p - p_0) + \frac{1}{\exp\left(\frac{\beta |p-p_0|^2}{2} + \alpha\right) - 1}$$

where $m_0 \geq 0,\ \beta \in (0,\infty],\ 0 \leq \alpha < \infty$ and $\alpha \cdot m_0 = 0,\ p_0 \in \mathrm{R}^3$.

$m_0 > 0$, Bose-Einstein condensation

Scenario for the formation of Bose-Einstein condensates.

- D.V. Semikov and I.I. Tkachev. (1995)
- R. Lacaze, P. Lallemand, Y. Pomeau and S. Rica. (2001)
- H. Spohn. (2010)

Numerical simulations combined with physical arguments suggest that:

(1) The **formation** of Bose-Einstein **condensates** takes place by means of a **blow-up** of the solutions of Nordheim equation.

(2) Blow-up is self-similar. (Second kind self-similarity). At the blow-up time $F(t,p)$ develops an integrable power law singularity.

(3) The number of particles at the condensate begins to increase as a power law after the blow-up time.

Isotropic Nordheim equation.

$$f(t,\epsilon) = F(t,p) \ , \ \epsilon = \frac{|p|^2}{2}$$

$$\partial_t f_1 = \frac{8\pi^2}{\sqrt{2}} \int_0^\infty \int_0^\infty q(f) W d\epsilon_3 d\epsilon_4$$

where:

$$W = \frac{\min\{\sqrt{\epsilon_1}, \sqrt{\epsilon_2}, \sqrt{\epsilon_3}, \sqrt{\epsilon_4}\}}{\sqrt{\epsilon_1}} \ , \ \epsilon_2 = \epsilon_3 + \epsilon_4 - \epsilon_1$$

$$q(f) = q_3(f) + q_2(f)$$

$$q_3(f) = f_3 f_4 (f_1 + f_2) - f_1 f_2 (f_3 + f_4)$$

$$q_2(f) = f_3 f_4 - f_1 f_2$$

Reformulation of the equation using the particle density in the energy space:

$$g(t,\epsilon) = 4\pi\sqrt{2\epsilon}\, f(t,\epsilon)$$

We can rewrite the equation using the density g :

$$\partial_t g_1 = 32\pi^3 \int_0^\infty \int_0^\infty q\left(\frac{g}{4\pi\sqrt{2\epsilon}}\right)\Phi d\epsilon_3 d\epsilon_4$$

$$\Phi = \min\{\sqrt{\epsilon_1}, \sqrt{\epsilon_2}, \sqrt{\epsilon_3}, \sqrt{\epsilon_4}\} \ , \ \ \epsilon_2 = \epsilon_3 + \epsilon_4 - \epsilon_1$$

Main result: **Finite time blow-up**.

Theorem *Let* $M > 0,\ E > 0,\ \nu > 0,\ \gamma > 3$. *There exist* $\rho = \rho(M,E,\nu) > 0,\ K^* = K^*(M,E,\nu) > 0,\ T_0 = T_0(M,E)$ *and a numerical constant* $\theta_* > 0$ *independent of* $M,\ E,\ \nu$ *such that for any* $f_0 \in L^\infty(\mathrm{R}^+;(1+\epsilon)^\gamma)$ *satisfying*

$$4\pi\sqrt{2}\int_{\mathrm{R}^+} f_0(\epsilon)\sqrt{\epsilon}\,d\epsilon = M\,,\ 4\pi\sqrt{2}\int_{\mathrm{R}^+} f_0(\epsilon)\sqrt{\epsilon^3}\,d\epsilon = E$$

$$\int_0^R f_0(\epsilon)\sqrt{\epsilon}\,d\epsilon \geq \nu R^{\frac{3}{2}},\ 0 < R \leq \rho,\ \int_0^\rho f_0(\epsilon)\sqrt{\epsilon}\,d\epsilon \geq K^*(\rho)^{\theta_*}$$

there exists a unique mild solution $f \in L^\infty_{loc}([0,T_{\max});L^\infty(\mathrm{R}^+;(1+\epsilon)^\gamma))$ *defined for a maximal existence time* $T_{\max} < T_0$. *The solution f satisfies:*

$$\lim \sup_{t\to T_{\max}^-} \|f(\cdot,t)\|_{L^\infty(\mathrm{R}^+)} = \infty$$

Bosonic Nordheim (-Boltzmann) equation vs. the classical Boltzmann equation.

$$\partial_t F_1 = \int_{R^3}\int_{R^3}\int_{R^3} q_2(F) M d^3p_2 d^3p_3 d^3p_4 \ , \ p_1 \in R^3 \ , \ t > 0$$

$$F_1(0,p) = F_0(p) \ , \ p_1 \in R^3 \ , \quad \epsilon = \frac{|p|^2}{2}$$

$$q_2(F) = F_3F_4 - F_1F_2$$

$$M = M(p_1,p_2;p_3,p_4) = \delta(p_1 + p_2 - p_3 - p_4)\delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4)$$

Global existence of solutions for the classical homogeneous Boltzmann equation (**Carleman**, **1933**).

Condensate formation.

Theorem *Let* $M > 0,\ E > 0,\ \nu > 0,\ \gamma > 3$. *There exist* $\rho = \rho(M,E,\nu) > 0,\ K^* = K^*(M,E,\nu) > 0,\ T_0 = T_0(M,E)$ *and a numerical constant* $\theta_* > 0$ *independent of* $M,\ E,\ \nu$ *such that for any* $f_0 \in L^\infty(\mathrm{R}^+;(1+\epsilon)^\gamma)$ *satisfying*

$$4\pi\sqrt{2}\int_{\mathrm{R}^+} f_0(\epsilon)\sqrt{\epsilon}\,d\epsilon = M\,,\ 4\pi\sqrt{2}\int_{\mathrm{R}^+} f_0(\epsilon)\sqrt{\epsilon^3}\,d\epsilon = E$$

$$\int_0^R f_0(\epsilon)\sqrt{\epsilon}\,d\epsilon \geq \nu R^{\frac{3}{2}},\ 0 < R \leq \rho,\ \int_0^\rho f_0(\epsilon)\sqrt{\epsilon}\,d\epsilon \geq K^*(\rho)^{\theta_*}$$

there exists a weak solution $f \in L^\infty_{loc}([0,T_{\max});L^\infty(\mathrm{R}^+;(1+\epsilon)^\gamma))$ *globally defined in time. There exists* $T_{con} > 0$ *such that:*

$$\int_0^R f(t,\epsilon)\sqrt{\epsilon}\,d\epsilon > 0 \text{ if } t > T_{con}$$

Proof of the results.(Difficult to compute explicit solutions):

Weak formulation:

$$\partial_t\left(\int_{\mathrm{R}^+} g\varphi d\epsilon\right) = \int_{\mathrm{R}^+} g\partial_t\varphi d\epsilon + \frac{1}{2^{\frac{5}{2}}}\int_{\mathrm{R}^+}\int_{\mathrm{R}^+}\int_{\mathrm{R}^+} \frac{g_1 g_2 g_3 \Phi}{\sqrt{\epsilon_1\epsilon_2\epsilon_3}} Q_\varphi d\epsilon_1 d\epsilon_2 d\epsilon_3 +$$

$$+\frac{\pi}{2}\int_{\mathrm{R}^+}\int_{\mathrm{R}^+}\int_{\mathrm{R}^+} \frac{g_1 g_2 \Phi}{\sqrt{\epsilon_1\epsilon_2}} Q_\varphi d\epsilon_1 d\epsilon_2 d\epsilon_3 \quad , \; a.e. \; t \in [0,T]$$

where:

$$\Phi = \min\left\{\sqrt{\epsilon_1}, \sqrt{\epsilon_2}, \sqrt{\epsilon_3}, \sqrt{(\epsilon_1+\epsilon_2-\epsilon_3)_+}\right\}$$

$$Q_\varphi = \varphi(\epsilon_3) + \varphi(\epsilon_1+\epsilon_2-\epsilon_3) - 2\varphi(\epsilon_1)$$

Monotonicity estimate:

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} \Phi Q_\varphi \prod_{j=1}^{3} \frac{g_j d\epsilon_j}{\sqrt{\epsilon_j}} = \int_{(\mathbb{R}^+)^3} \Phi G_\varphi \prod_{j=1}^{3} \frac{g_j d\epsilon_j}{\sqrt{\epsilon_j}}$$

where:

$$G_\varphi(\epsilon_1,\epsilon_2,\epsilon_3) = \frac{1}{6}\sum_{\sigma\in S^3} H_\varphi(\epsilon_{\sigma(1)},\epsilon_{\sigma(2)},\epsilon_{\sigma(3)})\Phi(\epsilon_{\sigma(1)},\epsilon_{\sigma(2)};\epsilon_{\sigma(3)})$$

$$H_\varphi(x,y,z) = \varphi(z) + \varphi(x+y-z) - \varphi(x) - \varphi(y)$$

with:

$$\Phi = \min\{\sqrt{\epsilon_1}, \sqrt{\epsilon_2}, \sqrt{\epsilon_3}, \sqrt{\epsilon_4}\} \;,\; \epsilon_2 = \epsilon_3 + \epsilon_4 - \epsilon_1$$

$G_\varphi(\epsilon_1,\epsilon_2,\epsilon_3) = G_\varphi(\epsilon_{\sigma(1)},\epsilon_{\sigma(2)},\epsilon_{\sigma(3)})$ for any $\sigma \in S^3$

φ is convex $\Rightarrow G_\varphi(\epsilon_1,\epsilon_2,\epsilon_3) \geq 0$, φ is concave $\Rightarrow G_\varphi(\epsilon_1,\epsilon_2,\epsilon_3) \leq 0$

Estimating the concentration properties of g.

(A general Measure Theory result).

Some notation:

$$\mathrm{I}_k(b) = b^{-k}\left(\frac{1}{b},1\right] \ , \ \ k = 0,1,2,\ldots \ \ , \ \ b > 1$$

$$\mathrm{I}_k^{(E)}(b) = \mathrm{I}_{k-1}(b) \cup \mathrm{I}_k(b) \cup \mathrm{I}_{k+1}(b) \ \ , \ \ k = 0,1,2,\ldots$$

$$\mathrm{P}_b = \left\{ A \subset [0,1] : A = \bigcup_j \mathrm{I}_{k_j}(b) \text{ for some set } \{k_j\} \subset \{1,2,\ldots\} \right\}$$

$$A^{(E)} = \bigcup_{j=1}^{\infty} \mathrm{I}_{k_j}^{(E)}(b)$$

Lemma *Suppose that* $b > 1$. *Given* $0 < \delta < \frac{2}{3}$, *there exists* $\eta > 0$ *such that, for any* $g \in \mathrm{M}^{+}[0,1]$ *satisfying* $\int_{\{0\}} g d\epsilon = 0$, *at least one of the following statements is satisfied:*

(i) There exist an interval $\mathrm{I}_k(b)$ *such that:*

$$\int_{\mathrm{I}_k^{(E)}(b)} g d\epsilon \geq (1-\delta) \int_{[0,1]} g d\epsilon$$

(ii) There exist two sets $\mathrm{U}_1, \mathrm{U}_2 \in \mathrm{P}_b$ *such that* $\mathrm{U}_2 \cap \mathrm{U}_1^{(E)} = \varnothing$ *and:*

$$\min\left\{\int_{\mathrm{U}_1} g d\epsilon, \int_{\mathrm{U}_2} g d\epsilon\right\} \geq \eta \int_{[0,1]} g d\epsilon$$

Blow-up for general supercritical data.

Theorem *Suppose that* $f_0 \in L^\infty(\mathrm{R}^+;(1+\epsilon)^\gamma)$ *with* $\gamma > 3$. *Let us denote as* $M,\ E$ *the numbers:*

$$4\pi \int_0^\infty f_0(\epsilon)\sqrt{2\epsilon}\, d\epsilon = M \;,\;\; 4\pi \int_0^\infty f_0(\epsilon)\sqrt{2\epsilon^3}\, d\epsilon = E$$

Let us denote as $f \in L^\infty_{loc}([0,T_{\max});L^\infty(\mathrm{R}^+;(1+\epsilon)^\gamma))$ *the unique mild solution of the Nordheim equation where* $T_{\max}$ *is the maximal existence time. Suppose that:*

$$M > \frac{\zeta\left(\frac{3}{2}\right)}{\left(\zeta\left(\frac{5}{2}\right)\right)^{\frac{3}{5}}}\left(\frac{4\pi}{3}\right)^{\frac{3}{5}} E^{\frac{3}{5}}$$

Then:

$$T_{\max} < \infty \;\text{ and }\; T_{con} < \infty$$

Main ideas in the Proof:

- An "smoothing" effect for Boltzmann equation.

$$\int_0^R g(\epsilon, t)\, d\epsilon \geq K R^{\frac{3}{2}} \quad \text{for } R \text{ small and } t \geq T_0(E, M).$$

- Entropy dissipation formula implies (in a weak form) that an amount of mass of order one concentrates near $\epsilon = 0$.

- We then apply the previous Theorem.

Qualitative information about the blow-up. (Not much rigorous information available).

Numerical simulations by Semikov-Tkachev and Lacaze-Lallemand-Pomeau-Rica suggest a self-similar behaviour:

$$f(t,\epsilon) = (T-t)^{-\alpha}\Phi\left(\frac{\epsilon}{(T-t)^{\beta}}\right) \ , \ \beta = \alpha - \frac{1}{2} \ , \ \nu = \frac{\alpha}{\beta}$$

$$f(T,\epsilon) \sim \frac{1}{\epsilon^{\nu}} \ , \ \nu = 1.234..$$

Theorem *(J. Bandyopadhyay, V):*

$$\limsup_{\epsilon \to 0}[\epsilon f(T,\epsilon)] = \infty$$

Weak Turbulence Equation.

$$\partial_t g_1 = 32\pi^3 \int_0^\infty \int_0^\infty q_3\left(\frac{g}{4\pi\sqrt{2\epsilon}}\right)\Phi d\epsilon_3 d\epsilon_4$$

$$\Phi = \min\{\sqrt{\epsilon_1}, \sqrt{\epsilon_2}, \sqrt{\epsilon_3}, \sqrt{\epsilon_4}\} \quad , \quad \epsilon_2 = \epsilon_3 + \epsilon_4 - \epsilon_1$$

$$q_3(g) = g_3 g_4 (g_1 + g_2) - g_1 g_2 (g_3 + g_4)$$

- Blow-up in finite time.
- Nonuniqueness (depending on how is the interaction between the condensate and the noncondensated part).
- Pulsating solutions: There exist measure solutions of this equation such that $g(\cdot, t) \rightharpoonup M\delta(\cdot)$ as $t \to \infty$ but $\int_{\{0\}} g(d\varepsilon, t) = 0$ for t large.
- Characterization of the long time asymptotics for general initial data:

"Almost" always $g(t, \cdot) \rightharpoonup M\delta(\cdot)$ as $t \to \infty$

A seemingly paradoxical situation:

Mass and energy conserved:

$$\partial_t\left(\int g(t,d\epsilon)\right) = \partial_t\left(\int \epsilon g(t,d\epsilon)\right) = 0$$

$$g(t,\cdot) \rightharpoonup M\delta(\cdot) \text{ as } t \to \infty$$

If $\int \epsilon g(0,d\epsilon) > 0$: where does the energy go?.

The only possibility is to $\epsilon = \infty$.

The **transfer of energy** towards infinity is **dominated** by the **interactions** between **two large particles** and **one small particle**.

The **equation** describing the **transfer of energy** towards infinity is:

$$\partial_t G(t,\omega) = \frac{1}{2}\int_0^{\omega} \frac{G(\omega-\xi)G(\xi)d\xi}{\sqrt{(\omega-\xi)\xi}} - \frac{G(\omega)}{\sqrt{\omega}}\int_0^{\infty} \frac{G(\xi)d\xi}{\sqrt{\xi}} - \frac{1}{2}\frac{G(\omega)}{\sqrt{\omega}}\int_0^{\omega}\left[\frac{G(\omega-\xi)}{\sqrt{\omega-\xi}} + \frac{G(\xi)}{\sqrt{\xi}}\right]d\xi + \int_0^{\infty}\frac{G(\omega+\xi)}{\sqrt{\omega+\xi}}\left[\frac{G(\omega)}{\sqrt{\omega}} + \frac{G(\xi)}{\sqrt{\xi}}\right]d\xi$$

Some interesting properties .

- Instantaneous condensation.

- Existence of self-similar solutions with constant energy.

(A. Kierkels, V)

We have obtained so far **two types of self-similar solutions**:

(1) *Energy conserving*:

$$G(t,\omega) = \frac{1}{t}\Phi\left(\frac{\omega}{\sqrt{t}}\right), \quad \Phi(\xi) \text{ decreases exponentially as } \xi \to \infty$$

(2) *Power law behaviour*:

$$G(t,\omega) = \frac{1}{t}\Phi\left(\frac{\omega}{t^{\frac{1}{\rho}}}\right), \quad \rho \in (1,2), \ \Phi(\xi) \sim \frac{1}{\xi^{\rho}} \text{ as } \xi \to \infty$$

If $\rho \in \left(\frac{1}{2}, 1\right]$ all the terms of the original cubic equation are expected to be relevant.

Conclusions

- Blow-up for all the "sufficiently concentrated" solutions of the isotropic bosonic Nordheim equation .

- Blow-up for all the supercritical data (Nordheim equation).

- Very few information about the behaviour of the solutions near the blow-up time.

- Self-similar behaviour for the solutions describing the long time asymptotics of the Weak Turbulence Equation.

Publications:

- M. Escobedo and JJLV. Comm. Math. Physics. 2014.
- M. Escobedo and JJLV. Invent. Mathem. 2015.
- M. Escobedo and JJLV. Memoirs AMS. 2015.
- J. Bandyopadhyay and JJLV. J. Math. Phys. 2015.
- A. Kierkels and JJLV. J. Stat. Phys. 2015.
- A. Kierkels and JJLV. J. Stat. Phys. 2016.
- A. Kierkels. Arxiv.

(3) Estimating the reaction rate.

Test function $\varphi(s) = (\frac{s}{R})^{\theta}$, $s \leq R$, $\varphi(s) = 1$ if $s \geq 1$.

Suppose that $\int_{\{0\}} g(\epsilon, t) d\epsilon = 0$ for any $t \in [0, T]$. Then:

$$B \int_0^T dt \int_{S_{R,\rho}} \prod_{m=1}^{3} g_m d\epsilon_m \leq \frac{R}{\rho^3} \left[2\pi \int_0^T dt \left(\int_{[0,1]} g(\epsilon) d\epsilon \right)^2 + M \right]$$

where $B > 0$, $\rho > 0$ small, $0 < R \leq \frac{1}{2}$ and:

$$\epsilon_+ = \max\{\epsilon_1, \epsilon_2, \epsilon_3\}\ ,\ \epsilon_- = \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$$

$$\epsilon_0 = \text{Intermediate value}$$

$$S_{R,\rho} = \left\{(\epsilon_1, \epsilon_2, \epsilon_3) \in [0, R]^3 : |\epsilon_0 - \epsilon_-| > \rho\epsilon_0\right\},\ \ 0 < R \leq 1,\ \ 0 < \rho < 1$$

An obstacle for the Proof:

There exist a family of stationary solutions of the isotropic cubic equation:

$$\partial_t g_1 = 32\pi^3 \int_0^\infty \int_0^\infty q_3\left(\frac{g}{4\pi\sqrt{2\epsilon}}\right)\Phi d\epsilon_3 d\epsilon_4$$

$$\Phi = \min\{\sqrt{\epsilon_1}, \sqrt{\epsilon_2}, \sqrt{\epsilon_3}, \sqrt{\epsilon_4}\} \ , \ \epsilon_2 = \epsilon_3 + \epsilon_4 - \epsilon_1$$

$$q_3(F) = F_3F_4(F_1 + F_2) - F_1F_2(F_3 + F_4)$$

Stationary solutions:

$$g_s(\epsilon) = M\delta(\epsilon - \epsilon_0) \ , \ \epsilon_0 > 0$$

If $g(t,\bullet)$ is close to one of these stationary solutions, the reaction would stop. (= A possible behaviour for global solutions).

General strategy for the Proof.

If g does not behave like a Dirac mass solution during most times, the reaction rate estimate would give a contradiction.

If g behaves like a Dirac mass solution during most times, it is possible to approximate the dynamics of the equation.

Lemma

An alternative. Either:

(a) During a significant fraction of the times in $[0, T_{\max}]$ we have that g is concentrated in a "dyadic" peak. (The position of the peak could change in time).

(b) During a significant fraction of the times in $[0, T_{\max}]$ we have that g is not concentrated in one small "dyadic" peak.

Both possibilities yield contradictions:

- Alternative (a) will imply a very fast transfer of particles towards the region $\epsilon \simeq 0$. This will contradict mass conservation.
- Alternative (b) contradicts the reaction rate estimate.

Solutions can be extended in time and they satisfy $\int_{\{0\}} g d\epsilon = 0$ as long as $\|f(t,\cdot)\|_\infty$ remains bounded. (Local existence result).

Therefore, the only alternative left is blow-up of $\|f(t,\cdot)\|_\infty$ as $t \to T_{\max}$.

Mass cannot move away from the region $\epsilon \simeq 0$.

Lemma *Suppose that* $\int_{[0,\frac{\rho}{2}]} g_0 d\epsilon \geq m_0 > 0,\ \int_0^\infty g_0 d\epsilon = M \geq m_0,$ $\int_0^\infty \epsilon g_0 d\epsilon = E > 0$ *where* $0 < \rho \leq 1.$ *There exists* $T_0 = T_0(M,E) > 0$ *independent on* ρ *and* $m_0,$ *such that for every solution* $f \in L^\infty([0,T_0];\mathrm{R}^+)$ *of the Nordheim equation such that* $f(\epsilon,0) = \frac{g_0(\epsilon)}{4\pi\sqrt{2\epsilon}}$ *we have*

$$\int_{[0,\rho]} g(\epsilon,t)d\epsilon \geq \frac{m_0}{4}$$

for $t \in [0,T_0],\ g = \frac{f}{4\pi\sqrt{2\epsilon}}$.

Proof: Monotonicity formula + Rough estimates of the quadratic (Boltzmann) terms.

Alternative (b) gives a contradiction:

Lemma *There exists* $\nu = \nu(\delta) > 0$ *independent on* R *and* ρ *such that, for any* $g \in \mathrm{M}^+[0,R]$ *satisfying* $\int_{\{0\}} g d\epsilon = 0$ *if the alternative (ii) in the Measure Theory Lemma takes place we have:*

$$\int_{S_{R,\rho}} \left[\prod_{m=1}^{3} g_m d\epsilon_m \right] \geq \nu \left(\int_{[0,R]} g d\epsilon \right)^3$$

Then, the reaction rate estimate implies:

$$\int_{A_T}\left(\int_{[0,R]} g d\epsilon\right)^3 dt \leq CR\left[\,2\pi\int_0^T dt\left(\int_{[0,1]} g(\epsilon)d\epsilon\right)^2 + M\,\right]$$

$A_T =$ Subset of $[0,T]$ where (ii) holds.

$$|A_T| \geq \frac{T}{2} \;\;,\;\; C = C(E, M, m_0)$$

$$\frac{T(m_0)^3}{2} \leq \tilde{C}R$$

Contradiction if R is small

Alternative (a) gives a contradiction:

If (a) takes place there is a very fast transfer of mass towards the region $\epsilon << 1$.

(This is due to the fact that reactions are faster for small ϵ and the lower estimate for the mass in the region of ϵ small).

This contradicts mass conservation.

Mass transfer is controlled using the adjoint equation. If the adjoint equation is chosen in the correct form, it is possible to see precisely that particles move approximately according to a transport equation:

$$\partial_t\left(\int g\varphi d\epsilon\right) = \int g(\partial_t\varphi + L(\varphi))d\epsilon$$

If the mass of g is concentrated near a peak, the adjoint equation can be approximated by a transport equation with small velocity.

Given $\theta_1 > 0,\ \theta_2 > 0$ such that $(1 - 2\theta_1 - \theta_2) > 0,$ let us assume that

there exists $\tilde{T}_0 \in [0, T_{\max}]$ such that

$$\int_0^{\tilde{T}_0} \chi_{A_\ell}(t) \left(\int_{I_{N(t)}^{(E)}(b_\ell, R_\ell)} g(t,\epsilon) d\epsilon \right)^2 dt = K_2 (R_\ell)^{1-\theta_2}\ ,\ K_2 = \left(\frac{\sqrt{2} - 1}{2} \right)$$

(Condition which guarantees that g is concentrated near a peak for a significant amount of time).

Lemma *If alternative (a) holds, there exists a function* $\varphi \in L^\infty([0,\tilde{T}_0], C^1(\mathrm{R}^+))$ *satisfying the following properties:*

(i) $0 \le \varphi(t,\epsilon) \le 1$ *for* $(t,\epsilon) \in [0,\tilde{T}_0] \times \mathrm{R}^+$.

(ii) $\varphi(t,\bullet)$ *is convex in* R^+ *for each* $t \in [0,\tilde{T}_0]$.

(iii) $\mathrm{supp}(\varphi(t,\bullet)) \subset \left[0, \frac{R_\ell}{4}\right]$ *for each* $t \in [0,\tilde{T}_0]$.

(iv) $\varphi(\epsilon,t) \ge \frac{1}{2}$ *for* $0 \le \epsilon \le \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)\frac{R_\ell}{8}$, $0 \le t \le \tilde{T}_0$.

(v) The following inequality holds for $0 \le t \le \tilde{T}_0$, $\epsilon \ge 0$:

$$\partial_t \varphi + \frac{\chi_{\mathrm{A}_\ell}(t)}{2^{\frac{3}{2}} R_\ell} \int\int_{U_\ell(t)} g_2 g_3 [\varphi(\epsilon_1 + \epsilon_3 - \epsilon_2) - \varphi(\epsilon_1)] d\epsilon_2 d\epsilon_3 \ge 0$$

$$U_\ell(t) = \left\{\epsilon_2 \le \epsilon_3,\ \epsilon_2, \epsilon_3 \in \mathrm{I}^{(E)}_{N(t)}(b_\ell, R_\ell)\right\}$$

Quantitative estimate for the blow-up time.

The previous arguments allow to estimate the measure of the set of times where the alternatives (a) and (b) hold. If the initial data is "concentrated enough" the sum of these measures is smaller than $T_0 = T_0(E, M)$. Therefore $T_{\max} < T_0$.