EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

Macroscopic aspects of the BCS-theory of superconductivity

Christian HAINZL

(Universität Tübingen)

Spohn-Feier, Minga

The critical temperature of a general many-particle system is associated with the following two-particle operator, corresponding to the linearized BdG-equation,

$$M_T + V \; : \; L^2(\mathbb{R}^d \times \mathbb{R}^d) \to L^2(\mathbb{R}^d \times \mathbb{R}^d), \quad d = 2, 3$$

$$M_T \alpha(x,y) = \frac{\mathfrak{h}_x + \mathfrak{h}_y}{\tanh \frac{\mathfrak{h}_x}{2T} + \tanh \frac{\mathfrak{h}_y}{2T}} \alpha(x,y)$$

$$V\alpha(x,y) = V(x-y)\alpha(x,y) \quad \text{superfluidity}$$
$$V\alpha(x,y) = V(x,y)\alpha(x,y) \quad \text{superconductivity},$$

$M_T + V$ ist the second derivative of the BCS-functional.

Formally we consider the two-body linear gap-equation

$$(M_T + V)\alpha = 0.$$

This is only formal, because $M_T + V$ has only essential spectrum.

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We consider particles (a) in small, slowly varying bounded external magnetic A and electric W potential, resp. (b) in a small, constant magnetic field $\mathbf{B}$:

(a)

$$\mathfrak{h} = (-i\nabla + hA(hx))^2 + h^2 W(hx) - \mu$$

(b)

$$\mathfrak{h} = \left(-i\nabla + \frac{\mathbf{B}}{2} \wedge x\right)^2 - \mu$$

$h \simeq \sqrt{B}$ is a **small parameter**

μ chemical potential, T temperature

Critical temperature

We define the critical tempemperature $T_c(h)$, $T_c(B)$ as the parameter T, which satisfies

$$\inf \sigma(M_T + V) = 0$$

How does the critical temperature $T_c(h)$, depend on A, W, respectively $T_c(B)$ depend on B?

Idea: We can handle it in the translation-invariant case $W = A = \mathbf{B} = 0$, afterwards we "perturb in h, B".

Difficulties:

- M_T is an *ugly* symbol.
- $\mathbf{B} \wedge x$ is not a bounded perturbation
- the components of $(-i\nabla + \mathbf{B} \wedge x)$ do *not* commute

We will deal with the Birman-Schwinger version

$$1 + V^{1/2} M_T^{-1} |V|^{1/2}.$$

T-I case $W = A = B = 0$

In the translation-invariant case $W = A = B = 0$ the symbol M_T is multiplication operator in momentum space.

$$\widehat{M_T\alpha}(p,q) = \frac{p^2 - \mu + q^2 - \mu}{\tanh \frac{p^2-\mu}{2T} + \tanh \frac{q^2-\mu}{2T}} \hat{\alpha}(p,q)$$

One has the algebraic inequality

$$M_T(p,q) \geq \frac{1}{2}\left(\frac{p^2-\mu}{\tanh \frac{p^2-\mu}{2T}} + \frac{q^2-\mu}{\tanh \frac{q^2-\mu}{2T}} \right) \geq 2T,$$

since

$$\frac{x}{\tanh \frac{x}{2T}} \geq 2T.$$

The task to recover the critical temperature is non-trivial, even in the T-I case. Let us first consider a toy model.

Simple toy model

Let us replace $M_T(p,q)$ by p^2+q^2+2T.

$$T_c : \inf \sigma(-\Delta_x - \Delta_y + 2T + V(x-y)) = 0$$

$$r = x - y \quad X = \frac{x+y}{2}$$

$$k = \frac{p-q}{2} \quad \ell = p + q$$

With $p = k + \ell/2$ and $q = k - \ell/2$, we get

$$p^2 + q^2 + 2T + V = 2k^2 + \ell^2/2 + 2T + V = -2\Delta_r + V(r) + 2T - \Delta_X/2,$$

hence

$$\inf \sigma(-\Delta_r + V(r)/2 + T_c) = 0 \quad \Leftrightarrow T_c = -e_0,$$

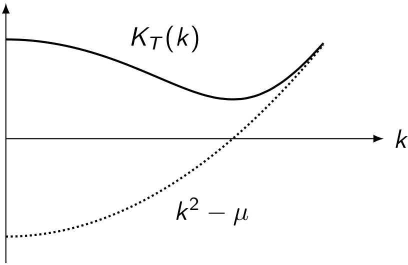
where e_0 is the smallest eigenvalue of $-\Delta_r + V/2$.

T_c is given by the one-particle operator for $\ell = 0$.

$M_T + V$ for $\ell = 0$ [HHSS]

At $\ell = 0$

$$M_T(k + \ell/2, k - \ell/2) = K_T(k) = \frac{k^2 - \mu}{\tanh((k^2 - \mu)/2T)}$$



For $\ell = 0$ one gets the one-particle operator

$$K_T + V(r): \ L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d).$$

Critical temperature: Since the operator $K_T + V$ is **monotone** in T, there exists unique $0 \le T_c < \infty$ such that

$$\inf \sigma(K_{T_c} + V) = 0,$$

respectively 0 is the lowest eigenvalue of $K_{T_c} + V$.

T_c is the critical temperature for the *effective* one particle system, if one reduces to translation-invariant states ($\ell = 0$).

[HHSS] C. Hainzl, E. Hamza, R. Seiringer, J.P. Solovej, Commun. Math. Phys. **281**, 349–367 **(2008)**.

Known results about $K_T + V$

- $\lim_{T\to 0} \frac{p^2-\mu}{\tanh \frac{p^2-\mu}{2T}} = |p^2-\mu|$, hence

$$T_c > 0 \textbf{ iff } \inf \sigma(|p^2-\mu| + V) < 0$$

- $\frac{1}{|p^2-\mu|}$ has same type of singularity as $1/p^2$ in $2D$ [S].
- In [FHNS, HS08, HS16] we classify V's such that $T_c > 0$. (E.g. $\int V < 0$ is enough)
- In [LSW] shown that $|p^2-\mu| + V$ has ∞ many eigenvalues if $V \leq 0$.
- the operator appeared in terms of scattering theory [BY93]

[FHNS] R. Frank, C. Hainzl, S. Naboko, R. Seiringer, Journal of Geometric Analysis, **17**, No 4, 549-567 (2007)

[HS08] C. Hainzl, R. Seiringer, Phys. Rev. B, **77**, 184517 (2008)

[HS16] C. Hainzl, R. Seiringer, J. Math. Phys. **57** (2016), no. 2, 021101

[BY93] Birman, Yafaev, St. Petersburg Math. J. 4, 1055-1079 (1993)

[LSW] A. Laptev, O. Safronov, T. Weidl, Nonlinear problems in mathematical physics and related topics I, pp. 233-246, Int. Math. Ser. (N.Y.), Kluwer/Plenum, New York (2002)

[S] B. Simon, Ann. Phys. 97, 279-288 (1976)

Lemma (FHSS12)

Let the 0 eigenvector of $K_{T_c} + V$ be non-degenerate. Then

(a)

$$M_{T_c} + V \gtrsim -\Delta_X$$

(b)

$$\inf \sigma(M_{T_c} + V) = 0$$

meaning T_c for the two-particle system is determined by the one-particle operator $K_T + V$ at $\ell = 0$.

The proof of (a) is non-trivial, because

$$M_T(k + \ell/2, k - \ell/2) \not\geq M_T(k,k) = K_T(k).$$

Recall in the toy-model this did hold

$$p^2 + q^2 + 2T = 2k^2 + \frac{1}{2}\ell^2 + 2T \geq 2k^2 + 2T.$$

(a) only holds for $V = V(x-y)$, *not* for general $V(x,y)$.

[FHSS12] R.L. Frank, C. Hainzl, R. Seiringer, J.P. Solovej, J. Amer. Math. Soc. **25**, 667–713 (2012).

Proof [FHSS12]

$$\begin{aligned} M_{T_c}(k+\ell/2, k-\ell/2) + V(r) &\geq \frac{1}{2}\left(K_{T_c}(k+\ell/2) + K_{T_c}(k-\ell/2)\right) + V(r) \\ &= \frac{1}{2}\left(e^{ir\cdot\ell/2}K_{T_c}(k)e^{-ir\cdot\ell/2} + e^{-ir\cdot\ell/2}K_{T_c}(k)e^{ir\cdot\ell/2}\right) + V(r) \\ &= \frac{1}{2}\left(U_\ell[K_{T_c}+V]U_\ell^* + U_\ell^*[K_{T_c}+V]U_\ell\right) \\ &\geq \kappa\frac{1}{2}\left(U_\ell[1-|\alpha_*\rangle\langle\alpha_*|]U_\ell^* + U_\ell^*[1-|\alpha_*\rangle\langle\alpha_*|]U_\ell\right) \\ &\geq \kappa\left[1-\left|\int \cos(\ell\cdot r)|\alpha_*(r)|^2 dr\right|\right] \simeq c\ell^2 \end{aligned}$$

for small momenta ℓ,

$$(K_{T_c}+V)\alpha_* = 0.$$

For large ℓ this is easy to see.

The proof crucially depends on V being translation invariant.

The proof is significantly harder if magnetic field **B** is included.

Theorem

Let $V \leq 0$, then there are parameters $\lambda_0, \lambda_1, \lambda_2$, depending on V, μ, such that

(a) *[FHSS14], with $\mathfrak{h} = (-i\nabla + hA(hx))^2 + h^2 W(hx) - \mu$, one has*

$$T_c(h) = T_c - h^2 D_c + o(h^2),$$

where

$$D_c = \frac{1}{\lambda_2} \inf \sigma(\lambda_0(-i\nabla + 2A(x))^2 + \lambda_1 W),$$

the lowest eigenvalue of the linearized Ginzburg-Landau *operator, A, W bounded.*

(b) *[FHL17], with $\mathfrak{h} = (-i\nabla + \frac{\mathbf{B}}{2} \wedge x)^2 - \mu$, one has*

$$T_c(B) = T_c - \frac{\lambda_0}{\lambda_2} 2B + o(B),$$

where

$$2B = \inf \sigma\left((-i\nabla + \mathbf{B} \wedge x)^2\right).$$

The (magnetic) Laplace in the Ginzburg-Landau is a universal property.

[FHSS14] R. L. Frank, C. Hainzl, R. Seiringer, J P Solovej, Commun. Math. Phys. 342 (2016), no. 1, 189–216
[FHL17] R. L. Frank, C. Hainzl, E. Langmann, in preparation

Ingredients of the proof

We consider the Birman-Schwinger version and define

$$T_c(h), T_c(B): \ \inf \sigma(1 - |V|^{1/2} L_T |V|^{1/2}) = 0, \quad L_T = M_T^{-1}$$

Advantage: L_T can be expressed in terms of resolvents.

$$L_T = \frac{1}{2i\pi} \int_C \tanh \frac{z}{2T} \frac{1}{z - \mathfrak{h}_x} \frac{1}{z + \mathfrak{h}_y} dz = T \sum_{n \in \mathbb{Z}} \frac{1}{\mathfrak{h}_x - i\omega_n} \frac{1}{\mathfrak{h}_y + i\omega_n}$$

with $\omega_n = \pi(2n+1)T$.

In (b) [FHL17] extension to $A = \mathbf{B} \wedge x$. *Surprisingly hard*.

Main problem: the first two components in $-i\nabla + \mathbf{B} \wedge x$ do not commute.

[FHSS14] R. L. Frank, C. Hainzl, R. Seiringer, J P Solovej, Commun. Math. Phys. 342 (2016), no. 1, 189–216

[FHSS12] R.L. Frank, C. Hainzl, R. Seiringer, J.P. Solovej, J. Amer. Math. Soc. **25**, 667–713 (2012).

Strategy of proof of (b)

1. step: For minimizing $1 - |V|^{1/2} L_T |V|^{1/2}$ we can reduce to states of the form

$$\varphi_*(x-y)\psi\left(\frac{x+y}{2}\right),$$

$$(1 - |V|^{1/2} K_{T_c}^{-1} |V|^{1/2})\varphi_* = 0 \quad \Leftrightarrow \quad (K_{T_c} + V)\alpha_* = 0, \; \varphi_*(x) = |V|^{1/2}(x)\alpha_*(x)$$

2. step: Show

$$\frac{1}{z - \mathfrak{h}_B}(x,y) \simeq e^{-i\frac{\mathbf{B}}{2}\cdot x\wedge y} \frac{1}{z - \mathfrak{h}_0}(x-y)$$

to evaluate

$$\langle \varphi_*\psi | 1 - |V|^{1/2} L_T |V|^{1/2} | \varphi_*\psi \rangle = \langle \varphi_* | 1 - |V|^{1/2} K_T^{-1} |V|^{1/2} | \varphi_* \rangle \|\psi\|^2$$

$$+ \int F(Z) \langle \psi(X) | 1 - \cos(Z \cdot (-i\nabla_X + \mathbf{B} \wedge X)) | \psi(X) \rangle dZ =$$

$$\langle \varphi_* | |V|^{1/2} (K_{T_c}^{-1} - K_T^{-1}) |V|^{1/2} | \varphi_* \rangle \|\psi\|^2 + \int F(Z) \langle \psi | 1 - \cos(Z \cdot (-i\nabla_X + \mathbf{B} \wedge X)) | \psi \rangle dZ$$

$$\simeq \lambda_2 (T - T_c) + \lambda_0 \langle \psi | (-i\nabla_X + \mathbf{B} \wedge X)^2 | \psi \rangle$$

Hence

$$\lambda_2(T - T_c) + \lambda_0 2B \simeq 0$$

and

$$T = T_c(B) \simeq T_c - \frac{\lambda_0}{\lambda_2} 2B$$

The derivative

$$\frac{d}{dB} T_c(0) = -\frac{\lambda_0}{\lambda_2} 2$$

was calculated by Helfand, Werthamer [HW].

[HW] E. Helfand, N.R. Werthamer, Phys. Rev. 147, 288 (1966)

Gap-equation

The equation

$$(M_T + V)\alpha = 0$$

is in the physics literature rewritten via

$$\alpha = -L_T V \alpha$$

as

$$\Delta = -V L_T \Delta, \qquad \Delta = V\alpha. \tag{1}$$

As $T \to 0$ the symbol $L_T(p,q)$ has poles for $|p| = |q|$.

If V is TI-invariant then pairs can only form for $q = -p$, i.e., for *Cooper-pairs* with total momentum 0.

High-T_c-superconductors

But if V is more general, $V = V(x, y)$, or integral operator $V(x, y; x', y')$, then more general pairs can form [HL].

In particular pairs with $q = p$.

We suggest that such pairs form in high-T_c-superconductors

[HL] C. Hainzl, M. Loss, EPJ B (2017)

Paper with Herbert (and Hirokawa)

Title: Binding energy for hydrogen-like atoms in the Nelson model without cutoffs

..... we investigate the radiative corrections to the binding energy and prove upper and lower bounds which imply that

$$E_{\mathrm{bin}} = \frac{me^4 Z^2}{2}\big(1 + (e^2/6\pi^2)\big) + O(e^7 \ln e)$$

independent of the ultraviolet cutoff.

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Congratulations

HAPPY BIRTHDAY