Adiabatic Theorem for Many Body Quantum Systems

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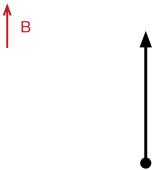

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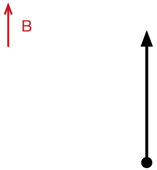
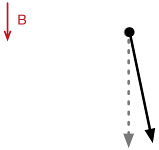


$\rightarrow$ $s = \infty$:



Anderson Catastrophe in Adiabatic Theory

For L independent spins with $H = \sum_x B_s \cdot \sigma_x$ the solution is a product state

$$\psi(s) = \otimes_x \psi_x(s) = \otimes_x(|\Omega_{x,s}\rangle + O(\varepsilon)).$$

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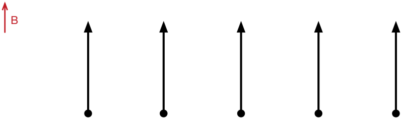

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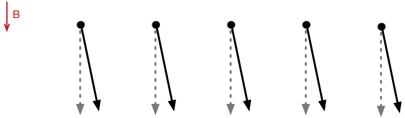

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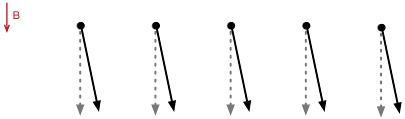

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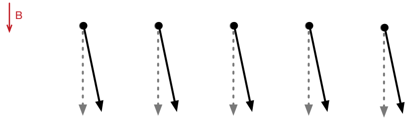

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We need an adiabatic theorem that survives the large volume limit!

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with $H_{X,s}$ acting on the subset X.
For extensive operator $B = \sum_X B_X$, define

$$||B||_{\text{loc}} := \sup_x || \sum_{X: X \ni x} B_X||.$$

- Local Hamiltonian: Hermitian op. with $||\cdot||_{\text{loc}} < C$ uniformly in L.
- Local observable: sitting at origin, independent of L.
- All bounds understood to be uniform in L

Local Adiabatic Theorem

Theorem

Let H_s be a family of local Hamiltonians for $0 \leq s \leq 1$ with following properties (all uniformly in s and size L)

- *finite range $H_{s,X} = 0$ for $\mathrm{diam}(X) > R$.*
- *unique ground state Ω_s*
- *gapped: $H_s \Omega_s^{\perp} \geq g > 0$.*
- *smooth $||\partial_s^k H_s||_{\mathrm{loc}} \leq C, \quad$ for $\quad k = 0, \ldots, d+2$*
- *smoothly start $\partial_s^k H_0 = 0$ for $k = 0, \ldots, d+2$ for $s = 0$.*

Then the solution $\psi(s)$ of the Schrödinger equation satisfies

$$|\langle \psi(s) | O | \psi(s) \rangle - \langle \Omega_s | O | \Omega_s \rangle| \leq C\varepsilon,$$

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Extension: isolated spectral patch instead of unique GS.

Three Key Ingredients

1. Construction of local dressing transformation.
2. Quasi-adiabatic continuation of Hastings and Wen [PRB 2005]
3. Lieb-Robinson bounds [CMP 1972]

Dressing transformation[1]

We construct $U_{s,n} = e^{-iA_{s,n}}$ with $||A||_{\mathrm{loc}} = O(\varepsilon)$ such that $\phi_{s,n} = U_{s,n}\Omega_s$ is a solution of

$$\varepsilon\partial_s\phi_s = -i(H_s + Y_{n,s})\phi_s, \quad \phi_0 = |\Omega_0\rangle$$

with $||Y_n||_{\mathrm{loc}} = O(\varepsilon^n)$.

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Dressing properties:

- $A_{s,n}$ is determined recursively
- Local in Space: $U_{s,n}$ produces correlations among $||A||^d_{\mathrm{loc}}$ sites.
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When $n \geq d + 2$, this solves our problem

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Sketch of the perturbative argument

Why does it help

- $O(s,s')$ is Heisenberg evolution from s' to s of local observable O.
- View evolution as small perturbation of the one generated by $H_s + Y_{n,s}$.
- Duhamel:

$$\langle \psi(s)|O|\psi(s)\rangle = \langle \phi_s|O|\phi_s\rangle + \frac{i}{\varepsilon}\int_0^s \langle \phi_{s'}|[Y_{n,s'}, O(s,s')]\phi_{s'}\rangle \mathrm{d}s'.$$

- LR: $O(s,s')$ supported in a ball of radius $\varepsilon^{-1}(s-s')$. Hence

$$|\langle \phi_{s'}|[Y_{n,s'}, O(s,s')]\phi_{s'}\rangle| \leq C\varepsilon^{n-d}.$$

Upshot: Using $n = d+2$ and $\langle \phi_s|O|\phi_s\rangle - \langle \Omega_s|O|\Omega_s\rangle = \mathcal{O}(\varepsilon)$,

$$\langle \psi(s)|O|\psi(s)\rangle - \langle \Omega_s|O|\Omega_s\rangle = \mathcal{O}(\varepsilon).$$

Construction of dressing I

Goal: construct $U_{s,n} = e^{-iA_{s,n}}$ with $||A||_{\mathrm{loc}} = O(\varepsilon)$ s.t. $\phi_{s,n} = U_{s,n}\Omega_s$ solves

$$\varepsilon \partial_s \phi_s = -i(H_s + Y_{n,s})\phi_s, \quad \phi_0 = |\Omega_0\rangle$$

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Simplifications:

- Drop s and n. We do $n = 1$.
- Assume $H_s\Omega_s = 0$.
- Multiply left and right by U^*. Set $\widetilde{Y} = U^* Y U$.

$\Rightarrow$

$$\epsilon U^*(U\Omega)' = -i(U^* H U + \widetilde{Y})\Omega$$

Construction of dressing II

Construct $U = e^{-i\epsilon A}$ with quasilocal A s.t.

$$\epsilon U^*(U\Omega)' = -i(U^* H U + \widetilde{Y})\Omega$$

ϵ^0: $\quad 0 = H\Omega \quad$ (ok by assumption)

ϵ^1: $\quad \Omega' = -[A, H]\Omega \quad$ (Difficult $\rightarrow$ next slide)

$\epsilon^{>1}$: $\quad \widetilde{Y} \equiv i\epsilon U^* U' - (U^* H U - i[A, H])$

So, if we solve $\Omega' = -[A, H]\Omega$ by local Ham A, then $\widetilde{Y}$ is a local Ham with $||\widetilde{|}||_{loc} \sim \epsilon^2$.

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Theorem (Hastings-Wen, Bachmann-Nachtergaele-Sims)

There is local Ham K implementing parallel transport:

$$\Omega' = iK\Omega, \qquad \langle \Omega, K\Omega \rangle = 0$$

(basis of whole philosophy 'quantum phases': $\Omega_1 = \mathcal{T} e^{i\int ds K_s} \Omega_0$)
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We still have to solve

$$iK\Omega = -[A, H]\Omega \quad \Leftrightarrow \quad iK\Omega = HA\Omega$$

Construction of dressing IV

First Idea to solve $iK\Omega = HA\Omega$:

$$A\Omega = \frac{i}{H}K\Omega \approx -\int_0^\infty dt\, e^{itH}K\Omega = -\int_0^\infty dt\, e^{itH}Ke^{-itH}\Omega$$

Comments

- $\frac{1}{H}K\Omega$ well-defined by $K\Omega \in \Omega^\perp = \chi[H \geq g]$.
- In general regularization needed for integral.
- $\tau_t(K) := e^{itH}Ke^{-itH}$ is quasilocal by Lieb-Robinson, but support grows $||\tau_t(K)||_{\text{loc}} \sim t$.
- Hence thus obtained

$$A \equiv \int_0^\infty dt\, \tau_t(K)$$

is not quasi-local (nor even well-defined, in fact)

Construction of dressing V

Second Idea to solve $K\Omega = HA\Omega$: Use spectral gap to write

$$iK\Omega = HA\Omega = f(H)A\Omega,$$

with

- $f(x) = x$ for $x \geq$ gap.
- $x \mapsto F(t) = \frac{i}{f(x)}$ is smooth and $\hat{F}$ decays rapidly.

Then

$$A\Omega = \frac{1}{f(H)}K\Omega = \int dt\, \hat{F}(t)e^{itH}K\Omega = \int dt\, \hat{F}(t)\tau_t(K)\Omega$$

Thus obtained

$$A \equiv \int dt\, \hat{F}(t)\tau_t(K)$$

is quasi-local by Lieb-Robinson+rapid decay of $\hat{F}$. This solves our issue!

Application: Validity of Linear response

Setting: An adiabatically switched on perturbation

$$H = H_{\mathrm{i}} + \alpha e^{\epsilon t} V, \qquad t \in (-\infty, 0]$$

with H_{i} and V local Hamiltonians and ψ_{i} ground state of H_{i}.
Linear response for local observable J:

$$\chi_{J,V} := \lim_{L\to\infty} \lim_{\alpha\to 0} \lim_{\epsilon\to 0} \frac{\langle \psi_t | J | \psi_t \rangle - \langle \psi_{\mathrm{i}} | J | \psi_{\mathrm{i}} \rangle}{\alpha}$$

'validity of linear response' is for us 1) existence of these limits and 2) the equality

$$\chi_{J,V} = i \int_0^\infty \langle \psi_{\mathrm{i}} | [V(t), J] | \psi_{\mathrm{i}} \rangle, \qquad V(t) = e^{itH_{\mathrm{i}}} V e^{-itH_{\mathrm{i}}}$$

Linear response: Kubo's formula

Theorem

If

$$H = H_{\mathrm{i}} + \beta V$$

is uniformly gapped for in a neighbourhood of $\beta = 0$, then for any local observable J

$$\chi_{J,V} := \lim_{\alpha \to 0} \lim_{\epsilon \to 0} \frac{\langle \psi_t | J | \psi_t \rangle - \langle \psi_{\mathrm{i}} | J | \psi_{\mathrm{i}} \rangle}{\alpha}$$

exists, uniformly in the volume.

Earlier results by Muller, Klein, Bru, Pedra for response smoothed in frequency.