Mathematatik 4 für Physik (Analysis 3) Zentralübung 10

Notiztitel 18.12.2012

TÜ3 heute 14:15 (Wolfgang Kinzner)

im MW 3501 (statt MW1401)

Blatt 9 A9.1: Hinweis (a) statt (b)

(b) statt (c)

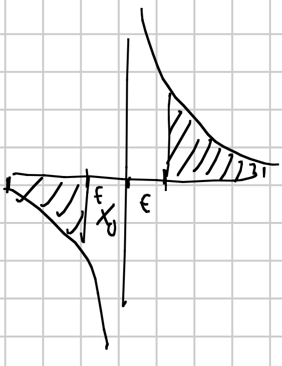
A9.3. existiert nicht

Cauchyscher Hauptwert

Definition des Hauptwertes: $f: \mathbb{R} \setminus \{x_0\} \to \mathbb{C}$ stetig.

Dann ist für $x_0 \in (a,b)$, $-\infty \leq a \leq \infty$

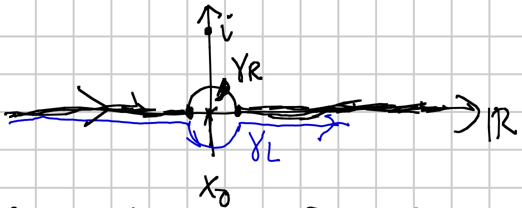
$$P\int_a^b f(x)\,dx = \lim_{\epsilon \to 0} \left(\int_a^{x_0-\epsilon} f(x)\,dx + \int_{x_0+\epsilon}^b f(x)\,dx \right)$$



falls der Limes existiert

Bsp: $$P\int_{-2}^{2} \frac{1}{x}\,dx = \lim_{\epsilon \to 0} \left(\left[\ln|x|\right]_{-2}^{-\epsilon} + \left[\ln x\right]_{\epsilon}^{2} \right.$$

$$= \lim_{\epsilon \to 0} \left(\ln\epsilon - \ln|-2| + \ln 2 - \ln\epsilon \right. = 0$$



Prinzipalwert $\gamma(t)=t$, $t\in[-2,2]$

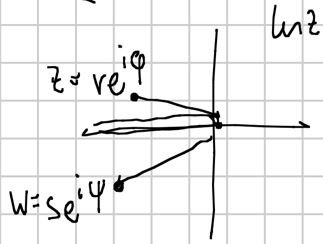
Beh: $\mathcal{P}\int_\gamma \frac{1}{z}\,dz = 0$, denn

unterer Grenze obere Grenze

$$\mathcal{R}\int_\gamma \frac{1}{z}\,dz = \int_{\gamma_R} \frac{1}{z}\,dz = -(\ln(-2)) + \ln 2 = -\ln 2 - i\pi + \ln 2 = -i\pi$$

$$\mathcal{L}\int_\gamma \frac{1}{z}\,dz = \int_{\gamma_L} \frac{1}{z}\,dz = -(\ln(-2) - 2\pi i) + \ln 2 = -\ln 2 + i\pi + \ln 2 = +i\pi$$

$$\mathcal{P}\int_\gamma \frac{1}{z}\,dz = \frac{1}{2}\left(\mathcal{R}\int_\gamma \frac{1}{z}\,dz + \mathcal{L}\int_\gamma \frac{1}{z}\,dz\right) = \mathcal{R}\int_\gamma \frac{1}{z}\,dz + i\pi \underbrace{\mathrm{Res}_0\left(\frac{1}{z}\right)}_{1} = 0$$



$\varphi = \pi - \epsilon$ $\quad \ln z = \ln r + i(\pi - \epsilon)$

$\psi = -\pi + \epsilon$ $\quad \ln w = \ln s + i(-\pi + \epsilon) \xrightarrow{\epsilon \to 0} \ln s - i\pi$

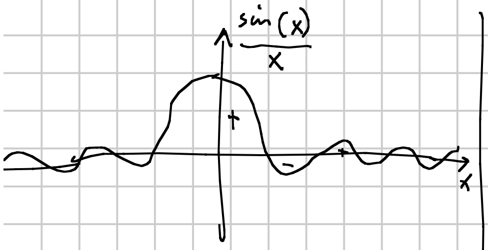
Vorlesung: Ist $f : [a,b] \setminus \{x_0\} \to \mathbb{C}$ holomorph, so ist mit $\gamma(t) = t$, $t \in [a,b]$

$$\boxed{\mathcal{P}\int_a^b f(x)\,dx = \mathcal{P}\int_\gamma f(z)\,dz = \mathcal{R}\int_\gamma f(z)\,dz + i\pi\,\mathrm{Res}_{x_0}(f)}$$

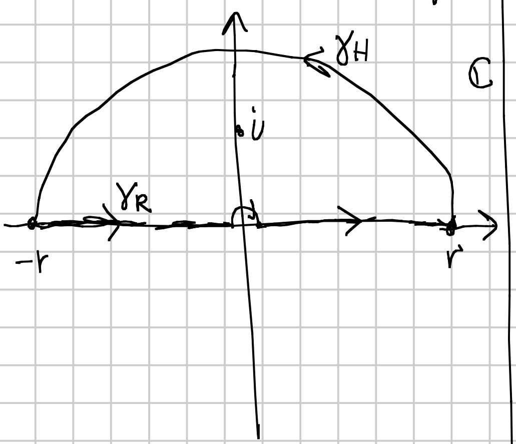
Anwendung: Berechne $\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \mathcal{P}\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$

(da $\frac{\sin x}{x} \to 1$ $(x \to 0)$)

$$= \mathcal{P}\int_{-\infty}^{+\infty} \frac{\mathrm{Im}(e^{ix})}{x}\,dx = \mathrm{Im}\,\mathcal{P}\int_{-\infty}^{+\infty} \frac{e^{ix}}{x}\,dx$$



$$P\int_{-\infty}^{+\infty} \frac{e^{ix}}{x}\,dx = \underbrace{R\int_{\gamma}}_{\gamma(t)=t,\ t\in\mathbb{R}} \underbrace{\frac{e^{iz}}{z}\,dz}_{=0} + i\pi \underbrace{\mathrm{Res}_0\left(\frac{e^{iz}}{z}\right)}_{=1}$$



$\mathbb{C}$

$$\underbrace{R\int_{\gamma} \frac{e^{iz}}{z}\,dz}_{\to 0 \text{ für } r\to\infty \Leftarrow} + \underbrace{\int_{\gamma_H} \frac{e^{iz}}{z}\,dz}_{\to 0 \text{ für } r\to\infty \text{ weil } \left|e^{i(x+iy)}\right| = e^{-y}} = 0$$

$$\Rightarrow P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = +i\pi\, \mathrm{Res}_0\left(\frac{e^{iz}}{z}\right) = +i\pi$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \operatorname{Im} P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = \pi \quad \blacksquare$$

Partialbruchzerlegung des cotangens

$$\cot z = \frac{\cos z}{\sin z}$$

$z \mapsto \pi\cot\pi z$ hat einfache Pole bei $n\in\mathbb{Z}$

$$\mathrm{Res}_n(\pi\cot\pi z) = \frac{\pi\cos(\pi n)}{\pi\sin'(\pi n)} = 1$$

Für $g_n(z) := \sum_{k=-n}^{n} \frac{1}{z-k}$ gilt:

$\pi\cot\pi z - g_n(z)$ hat keine Pole im Quadrat $Q_n = \left[-(n+\frac{1}{2}), n+\frac{1}{2}\right]^2 \subseteq \mathbb{C}$

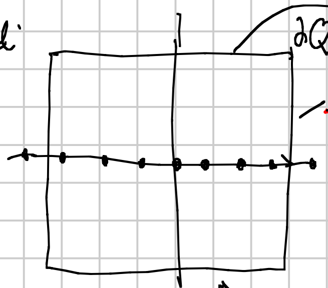
Satz: $\pi\cot\pi z = \lim_{n\to\infty} g_n(z)$

Lemma 1 $\pi\cot\pi\left[\left(n+\frac{1}{2}\right)+iy\right] = i\pi\tanh(\pi y)$

Beweis $\pi\cot\pi\left(\left(n+\frac{1}{2}\right)+iy\right) = -\pi\tan(i\pi y) = -\pi\frac{\sin i\pi y}{\cos i\pi y} = -\pi\frac{i\sinh\pi y}{\cosh\pi y} = -i\pi\tanh(\pi y)$

Lemma 2: $\left|\pi\cot\pi z\right| \leq C$ für $z\in\partial Q_n$, $n\in\mathbb{N}$ bel

Beweisskizze:



kgt gegen $i\pi$, stetig und 1-periodisch

$-i\tanh$, beschränkt durch π

kgt gegen $i\pi$

x fest: $\pi\cot\pi(x+iy) = i\pi\frac{e^{i\pi(x+iy)}+e^{-i(x+iy)\pi}}{e^{i\pi(x+iy)}-e^{-i(x+iy)\pi}} \begin{cases} \xrightarrow{y\to+\infty} -i\pi \\ \xrightarrow{y\to-\infty} +i\pi \end{cases}$

Es kann $C=4$ gewählt werden

Lemma 3 p,q Polynome, $\operatorname{grad} q \geq \operatorname{grad} p+2$, q habe keine ganzzahligen Nullstellen. Dann gilt

$$\sum_{n\in\mathbb{Z}}\frac{p(n)}{q(n)} = -\sum_{\{a,\,q(a)=0\}}\operatorname{Res}_a\left(\frac{p}{q}\right)\pi\cot(\pi a)$$

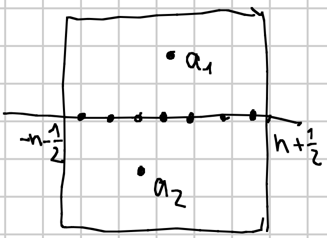
Beweis: n groß genug

$$\oint_{\partial Q_n}\frac{p(z)}{q(z)}\pi\cot\pi z\,dz = \sum_{\{a,\,q(a)=0\}}\operatorname{Res}_a\left(\frac{p}{q}\right)\pi\cot(\pi a) + \underbrace{\sum_{k=-n}^{n}\frac{p(k)}{q(k)}}_{\text{abs. kgt}}$$

$\Big\downarrow n\to\infty$

0

$\frac{p(z)}{q(z)} \sim \frac{1}{z^2} \to 0 \quad |z|\to\infty$



$$\sum_{k\in\mathbb{Z}} \frac{1}{1+k^2} = ?$$

$\underline{\text{Satz}}$: $\pi \cot \pi z = \lim\limits_{n\to\infty} \sum\limits_{k=-n}^{n} \frac{1}{z-k}$ für $z \notin \mathbb{Z}$

Beweis: $\displaystyle\sum_{k=-n}^{n} \frac{1}{w-k} = \frac{1}{2} \sum_{k=-n}^{n} \left(\frac{1}{w-k} + \frac{1}{w+k}\right) = \sum_{k=-n}^{n} \frac{w}{w^2-k^2}$

$$\lim_{n\to\infty} \sum_{k=-n}^{n} \frac{w}{w^2-k^2} = -\underbrace{\mathrm{Res}_{-w}\left(\frac{w}{w^2-z^2}\right)}_{+\frac{1}{2}} \underbrace{\pi\cot\pi(-w)}_{-\cot(\pi w)} - \underbrace{\mathrm{Res}_{w}\left(\frac{w}{w^2-z^2}\right)}_{-1/2} \pi\cot(\pi w)$$

$$= \pi \cot \pi w \quad \blacksquare$$