Mathematik 4 für Physik (Analysis 3) Zentralübung 9

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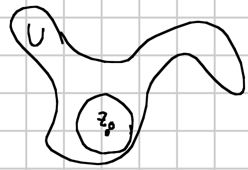
$f: U \to \mathbb{C}$ holomorph

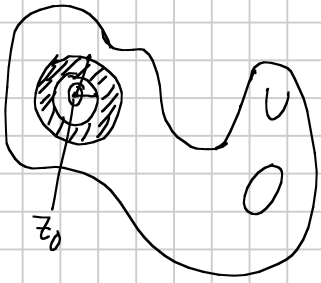


- $K_r(z_0) \subseteq U \Rightarrow$ Potenzreihe von f in z_0 hat Kgzradius $\geq r$
- $K_{r,R}(z_0) \subseteq U \Rightarrow$ Laurentreihe von f in z_0 konvergiert auf $K_{r,R}(z_0)$

$$c_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz \; : \; f(z) = \sum_{n\in\mathbb{Z}} c_n (z-z_0)^n$$

für $z \in K_{r,R}(z_0)$

Residuum für $n=-1$

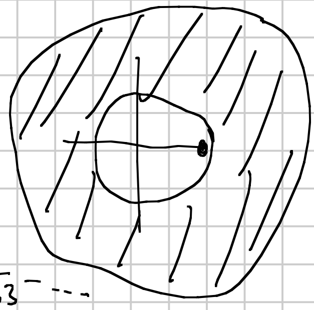


Laurentreihen:

- $$\frac{1}{1-z} = 1 + z + z^2 + \dots \quad \text{auf } K_1(0)$$

$$\frac{1}{1-z} = -\frac{1}{z}\,\frac{1}{1-\frac{1}{z}} = -\frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \dots\right) = -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \dots$$

auf $K_{1,\infty}(0)$



- $$\frac{1}{(1-z)^2} \overset{\text{PBZ}}{=} \frac{\overset{=0}{A}}{1-z} + \frac{\overset{=1}{B}}{(1-z)^2} = \frac{1}{1-z}\cdot\frac{1}{1-z} = \dots \quad \text{Cauchyproduktformel}$$

$$= \frac{d}{dz}\frac{1}{1-z} \begin{cases} = 1 + 2z + 3z^2 + \dots & \text{auf } K_1(0) \\ = \frac{1}{z^2} + \frac{2}{z^3} + \frac{3}{z^4} + \dots & \text{auf } K_{1,\infty}(0) \end{cases}$$

Mit PBZ erhält man Laurentreihen für alle rationalen Funktionen

- $z^5 \sin\frac{1}{z} = z^5\left(\frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} \mp \cdots\right)$

$$= \underbrace{z^4 - \frac{1}{3!}z^2 + \frac{1}{5!}}_{\text{Nebenteil}} \underbrace{- \frac{1}{7!}\frac{1}{z^2} \pm \cdots}_{\text{Hauptteil}}$$

Residuen

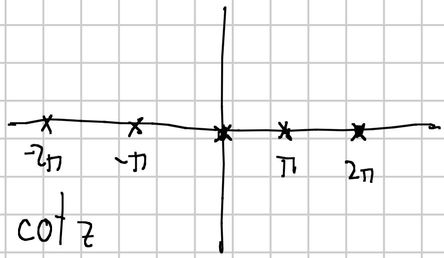
Wichtigste Regel: f,g holomorph um z_0, z_0 einfache Nullstelle von g ($g(z_0)=0$, $g'(z_0)\neq 0$)

$$\boxed{\operatorname{Res}_{z_0}\left(\frac{f}{g}\right) = \frac{f(z_0)}{g'(z_0)}}$$ insbesondere ist $$\boxed{\operatorname{Res}_{z_0}\left(\frac{f(z)}{z-z_0}\right) = f(z_0)}$$

Bsp.: • $\cot z = \frac{\cos z}{\sin z}$. Bei $z_k = \pi k$, $k\in\mathbb{Z}$ einfache Pole:

$\sin \pi k = 0 \quad \sin'(\pi k) = \cos \pi k = (-1)^k$

$$\operatorname{Res}_{z_k}(\cot) = \frac{\cos(z_k)}{\sin'(z_k)} = 1$$



$$\lim_{N\to\infty}\sum_{k=-N}^{N} \frac{1}{z-z_k} \overset{?}{=} \cot z$$

- $\frac{1}{\sin^2 z}$ Pol zweiter Ordnung bei 0. $k=2$

$$\operatorname{Res}_0\left(\frac{1}{\sin^2 z}\right) = \frac{1}{(k-1)!}\lim_{z\to z_0} \underbrace{\left(\frac{d}{dz}\right)^{k-1}\left(\frac{z^k}{\sin^2 z}\right)}_{\frac{2z}{\sin^2 z} - \frac{z^2\, 2\cos z}{\sin^3 z}}$$ $\to 0$ für $z\to 0$

Einfacher: $$\frac{1}{\sin^2 z} = \frac{1}{\left(z - \frac{z^3}{3!} \pm\cdots\right)^2} = \frac{1}{z^2}\frac{1}{\left(1-\frac{z^2}{3!}\pm\cdots\right)^2} = \frac{1}{z^2}\frac{1}{1-\frac{1}{3} z^2 + O(z^4)}$$

$$= \frac{1}{z^2}\left(1 + \left(\frac{1}{3} z^2 + O(z^4)\right) + O(z^4)\right) = \frac{1}{z^2} + \frac{1}{3} + O(z^2)$$

$$\Rightarrow \operatorname{Res}_0\left(\frac{1}{\sin^2 z}\right) = 0$$

- Aber i.A. aufwändiger:

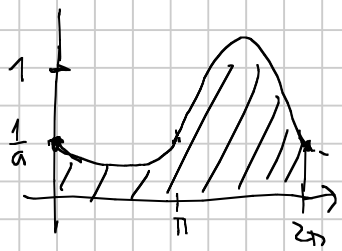
$$f(z) = \frac{1}{z^2(1-z)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{1-z} \qquad \begin{array}{|l} 1\cdot(1-z):\ C=1 \\ 1\cdot z^2:\ B=1 \end{array}$$

$$= \frac{Az(1-z) + B(1-z) + Cz^2}{z^2(1-z)} = \frac{Az - Az^2 + 1 - z + z^2}{\dots}, \text{ also } A = 1.$$

d.h. $\operatorname{Res}_0(f) = 1$

Residuenkalkül

Bsp 1. $\displaystyle\int_0^{2\pi} \frac{dt}{a+\sin t}, \quad a>1$



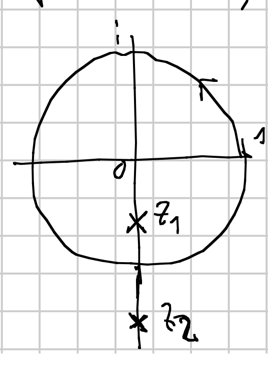
"Substitution" $z = e^{it}$. Eigentlich $\gamma(t) = e^{it}$, $\gamma'(t) = ie^{it}$

$$\int_0^{2\pi} \frac{dt}{a + \frac{1}{2i}\left(e^{it} - e^{-it}\right)} = \int_0^{2\pi} \frac{ie^{it}}{ie^{it}\left(a + \frac{1}{2i}e^{it} - \frac{1}{2i}\underbrace{e^{-it}}_{\frac{1}{\gamma(t)}}\right)}\,dt = \oint_\gamma \frac{1}{iz\left(a + \frac{z}{2i} - \frac{1}{2iz}\right)}\,dz$$

$$= \oint_\gamma \frac{2}{z^2 + 2iaz - 1}\,dz$$

Nullst. $z_{1/2} = \dfrac{-2ia \pm \sqrt{-4a^2+4}}{2} = -ia \pm i\sqrt{a^2-1} = i\left(\pm\sqrt{a^2-1} - a\right)$

PBZ $\dfrac{2}{z^2+2iaz-1} = \dfrac{\frac{-i}{\sqrt{a^2-1}}}{z-z_1} + \dfrac{\frac{i}{\sqrt{a^2-1}}}{z-z_2}$

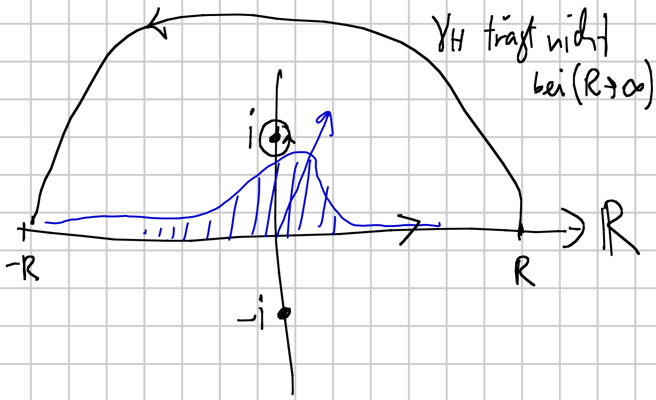


$|z_2| = a + \sqrt{a^2-1} > 1$

$|z_1| = a - \sqrt{a^2-1} = \dfrac{a^2 - a^2 + 1}{a + \sqrt{a^2-1}} = \dfrac{1}{|z_2|} < 1$

$$\Rightarrow \int_0^{2\pi} \frac{dt}{a+\sin t} = \oint_\gamma \frac{2}{z^2+2iaz-1}\,dz = 2\pi i \operatorname{Res}_{z_1}\left(\frac{2}{z^2+2iaz-1}\right) = \frac{2\pi}{\sqrt{a^2-1}} \quad \boxed{}$$

Beispiel 2: $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$



$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$$

$$\operatorname{Res}_i(f) = \frac{1}{2i}, \quad \operatorname{Res}_{-i}(f) = \frac{-1}{2i}$$

$$\int_{-R}^{R} \frac{1}{1+x^2}\,dx = 2\pi i \operatorname{Res}_i(f) - \underbrace{\int_{\gamma_H} f(z)\,dz}_{\to 0} \qquad \text{(Residuensatz)}$$

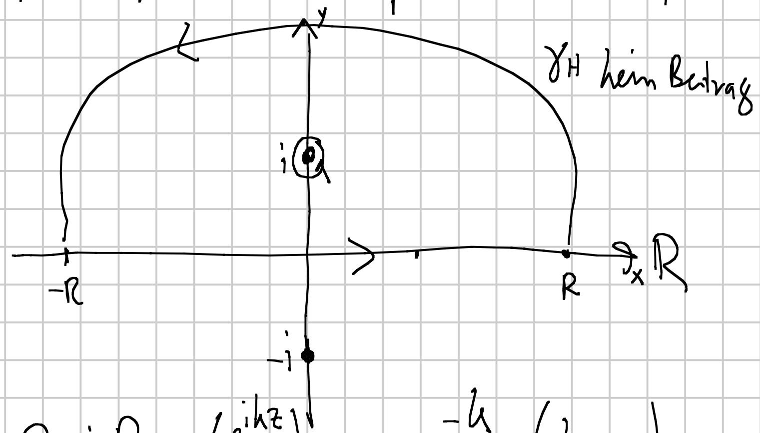
$$\downarrow R\to\infty$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{2\pi i}{2i} - 0 = \pi \quad \big(= \operatorname{atan}(\infty) - \operatorname{atan}(-\infty)\big)$$

Beispiel 3: $F(k) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{1+x^2}\,dx \qquad k\in\mathbb{R}$

Für $k=0$: $F(0)=\pi$

Für $k>0$: $|e^{ikz}| = |e^{ik(x+iy)}| = e^{-ky}$. exponentiell klein für $y>0$



$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{1+x^2}\,dx = 2\pi i \operatorname{Res}_i\left(\frac{e^{ikz}}{1+z^2}\right) = \pi e^{-k} \quad (k>0)$$

Bemerkung: g einfache Nullst. bei z_0, d.h. $\mathrm{Res}_{z_0}\left(\frac{1}{g}\right) = \frac{1}{g'(z_0)}$

$$\mathrm{Res}_{z_0}\left(\frac{f}{g}\right) = f(z_0)\,\mathrm{Res}_{z_0}\left(\frac{1}{g}\right)$$

Für $k<0$

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{1+x^2}\,dx = -2\pi i\,\mathrm{Res}_{-i}\left(\frac{e^{ikz}}{1+z^2}\right) = -\pi\left(-e^{k}\right)$$

Insges. $$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{1+x^2}\,dx = \pi e^{-|k|}$$