Mathematik 3 für Physik (Analysis 2) Zentralübung 13

Notiztitel 15.07.2013

Bonus 32 sinnvoll bearb. Aufgaben + Vorrechnen

Probeklausur

A6

$$x + y + \sin z = 0$$
$$3\sin x - 2\tan y - z = 0$$

$y = -x - \sin z$, $\tilde{y}(x) = -x - \sin \tilde{z}(x)$

$z = \tilde{z}(x)$

$f: \mathbb{R}^3 \to \mathbb{R}^2$, $f(x,y,z) = \begin{pmatrix} x + y + \sin z \\ 3\sin x - 2\tan y - z \end{pmatrix}$,

Löse $f(x,y,z) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ nach y und z auf.

$f(0,0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $J_f(x,y,z) = \begin{pmatrix} 1 & 1 & \cos z \\ 3\cos x & -\frac{2}{\cos^2 y} & -1 \end{pmatrix} \in \mathbb{R}^{2\times 3}$

$$J_f(0,0,0) = \begin{pmatrix} 1 & 1 & 1 \\ 3 & -2 & -1 \end{pmatrix} \quad x \quad y \quad z$$

$g(x) = \begin{pmatrix} \tilde{y}(x) \\ \tilde{z}(x) \end{pmatrix}$ $\quad g: \mathbb{R} \to \mathbb{R}^2$

$$\frac{dg}{dx}(x) = -\begin{pmatrix} \frac{\partial f_1}{\partial y}(\,) & \frac{\partial f_1}{\partial z}(\,) \\ \frac{\partial f_2}{\partial y}(\,) & \frac{\partial f_2}{\partial z}(\,) \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial f_1}{\partial x}(x, \tilde{y}(x), \tilde{z}(x)) \\ \frac{\partial f_2}{\partial x}(x, \tilde{y}(x), \tilde{z}(x)) \end{pmatrix}$$

$$\frac{dg}{dx}(0) = -\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = -\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$$

d.h. $\tilde{y}'(0) = 4$, $\tilde{z}'(0) = -5$

(b) Lösungskurve $\gamma : \mathbb{R} \to \mathbb{R}^3$ $\quad \gamma(x) = \begin{pmatrix} x \\ \tilde{y}(x) \\ \tilde{z}(x) \end{pmatrix}$

$$\gamma'(0) = \begin{pmatrix} 1 \\ 4 \\ -5 \end{pmatrix} \quad \text{Normiert ergibt das} \quad \frac{1}{\sqrt{42}} \begin{pmatrix} 1 \\ 4 \\ -5 \end{pmatrix}.$$

$$\tilde{y}'(x) = -\left(\frac{\partial f_2}{\partial z}(x, \tilde{y}(x), \tilde{z}(x)) \frac{\partial f_1}{\partial x}(x, \tilde{y}(x), \tilde{z}(x)) + \frac{\partial f_1}{\partial z}(\dots) \frac{\partial f_2}{\partial x}(\dots) \right) \Big/ \det \begin{pmatrix} \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix} (\dots)$$

$$F(x,y,z) = 0 \in \mathbb{R}$$

$$\tilde{z}(x,y) \quad , \quad \underbrace{J\tilde{z}(x,y)}_{1 \times 2} = -\underbrace{\left(\frac{\partial F}{\partial z}(x,y,\tilde{z}(x,y)) \right)^{-1}}_{1 \times 1} \underbrace{\left(\frac{\partial F}{\partial x}(\dots) \quad \frac{\partial F}{\partial y}(\dots) \right)}_{1 \times 2}$$

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7. Lagrangemultiplikator

$P = (x_0, y_0, z_0) \in \mathbb{R}^3$ regulärer Punkt von $f \in C^1(\mathbb{R}^3, \mathbb{R})$, $f(P) = 0$.

$f(x,y,z) = 0$ lokal aufgelöst nach z ergibt $\tilde{z}(x,y)$ $\quad (\tilde{z}(x_0,y_0) = z_0)$

(a) $$\nabla \tilde{z}(x_0,y_0)^T = J\tilde{z}(x_0,y_0) = -\frac{\partial f}{\partial z}(x_0,y_0,z_0)^{-1} \left(\partial_x f(P) \quad \partial_y f(P) \right)$$

$$\nabla \tilde{z}(x_0,y_0) = \begin{pmatrix} -\frac{\partial_x f(P)}{\partial_z f(P)} \\ -\frac{\partial_y f(P)}{\partial_z f(P)} \end{pmatrix} \quad \text{mit } \partial_z f(P) \neq 0$$

(b) $h \in C^1(\mathbb{R}^3, \mathbb{R})$, sei $\tilde{h}(x,y) = h(x,y,\tilde{z}(x,y))$

$\tilde{h} \in C^1(\mathbb{R}^2, \mathbb{R})$. Sei (x_0, y_0) stationärer Punkt von $\tilde{h}$

Zeige, dass dann $\nabla h(P) = \lambda \nabla f(P)$.

Bew.: $\nabla\tilde{h}(x_0,y_0)=0$ bedeutet

$$0 = (\partial_x \tilde{h})(x_0,y_0) \stackrel{!}{=} \frac{d}{dx} h\left(x, y_0, \tilde{z}(x,y_0)\right)\Big|_{x=x_0} =$$

$$\left[= \lim_{h\to 0} \frac{\tilde{h}(x_0+h,y_0)-\tilde{h}(x_0,y_0)}{h} = \lim_{h\to 0} \frac{h(x_0+h,y_0,\tilde{z}(x_0+h,y_0)) - h(x_0,y_0,\tilde{z}(x_0,y_0))}{h}\right)$$

$$= \partial_x h(P)\cdot 1 + \partial_y h(P)\cdot 0 + \partial_z h(P) \frac{\partial \tilde{z}}{\partial x}(x_0,y_0)$$

$$= \partial_x h(P) - \partial_z h(P) \frac{\partial_x f(P)}{\partial_z f(P)}$$

$$0 = \partial_y \tilde{h}(x_0,y_0) = \frac{d}{dy} h\left(x_0, y, \tilde{z}(x_0,y)\right)\Big|_{y=y_0}$$

$$= \partial_y h(P) + \partial_z h(P) \frac{\partial \tilde{z}}{\partial y}(x_0,y_0) = \partial_y h(P) - \partial_z h(P) \frac{\partial_y f(P)}{\partial_z f(P)}$$

$$\left.\begin{aligned} \partial_x h(P) &= \frac{\partial_z h(P)}{\partial_z f(P)} \cdot \partial_x f(P) \\ \partial_y h(P) &= \frac{\partial_z h(P)}{\partial_z f(P)} \cdot \partial_y f(P) \\ \partial_z h(P) &= \frac{\partial_z h(P)}{\partial_z f(P)} \cdot \partial_z f(P) \end{aligned}\right\} \Rightarrow \quad \nabla h(P) = \lambda \nabla f(P), \quad \lambda = \frac{\partial_z h(P)}{\partial_z f(P)} \in \mathbb{R}.$$

$\ln = \log : \mathbb{R}^+ \to \mathbb{R}$ $\qquad \log_{10}(x) = \frac{\log(x)}{\log(10)} = \frac{\ln(x)}{\ln(10)}$

$\|x\| = \|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$ für $x \in \mathbb{R}^n$

Richtungsableitung: $f: \mathbb{R}^n \to \mathbb{R}$ Richtungsabl. von f Richtung $v \in \mathbb{R}^n$

$$x \in \mathbb{R}^n: \; \partial_v f(x) = \lim_{h\to 0} \frac{f(x+hv)-f(x)}{h} \underbrace{= \frac{d}{dt} f(x+tv)\Big|_{t=0}}_{\text{Ergänzung}}$$

Banachscher Fixpunktsatz: M vollst. metr. Raum $f: M \to M$ Kontraktion

$\Rightarrow \exists!\, x \in M$ s.d. $f(x) = x$

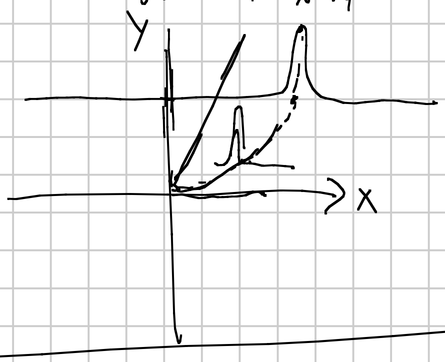
f Kontraktion: $\exists\, \alpha \in (0,1)$ s.d. $\forall x, y \in M \quad d(f(x), f(y)) \leq \alpha\, d(x,y)$.

Totale Differenzierbarkeit. $f: \mathbb{R}^n \to \mathbb{R}$ ist total differenzierbar in $x_0 \in \mathbb{R}^n$, wenn es eine $1 \times n$-Matrix A gibt, sodass f.a. $\Delta \in \mathbb{R}^n$

$$\frac{\| f(x+\Delta) - f(x) - A\Delta \|}{\|\Delta\|} \to 0$$

Wie bekomme ich das A: $\quad A = J_f(x) = \begin{pmatrix} \partial_1 f(x) & \cdots & \partial_n f(x) \end{pmatrix}$

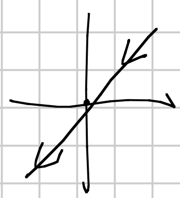
$$f(x,y) = xy\,\frac{x^2-y^2}{x^2+y^2}.$$

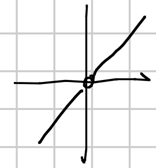


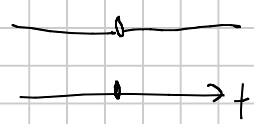
$$f\left(\frac{1}{n}, \frac{1}{n^2}\right) = 1 \neq 0 = f(0,0)$$

$$f(x,y) = \frac{xy}{x^2+y^2},\ f(0,0) = 0,\ t \mapsto f(tv_1, tv_2) = \frac{t^2 v_1 v_2}{t^2(v_1^2+v_2^2)} = \frac{v_1 v_2}{v_1^2+v_2^2} \neq 0 \text{ für } v_1, v_2 \neq 0$$

$$g(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$$







$$|g(x,y)| \leq \frac{\|(x,y)\|^2}{\|(x,y)\|} = \|(x,y)\|$$

$$t \mapsto g(t,t) = \frac{t^2}{\sqrt{2}\,|t|} = \frac{1}{\sqrt{2}}|t|$$