Seminar uncertainty relations
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Uncertainty relations and Fourier analysis

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1 Robertson-Schrödinger uncertainty relation

We discuss an uncertainty inequality for self-adjoint operators on a Hilbert space. In quantum mechanics self-adjoint operators play a fundamental role. The states of a quantum mechanical system are described as unit vectors in an appropriate Hilbert space $\mathcal{H}$ and the observable quantities are self-adjoint operators on $\mathcal{H}$. In this context $M(\mu_u) := \langle Au, u\rangle$ can be interpreted as the expectation value of A in the state u and $V(\mu_u) := ||(A - M(\mu_u))u||^2$ is a measure of the uncertainty of A in the state u.
The following theorem states that there is a positive lower bound for the product of two observables A and B in a state u as long as $\langle ABu, u\rangle \neq \langle BAu, u\rangle$.

Let A and B be densely defined operators on a Hilbert space $\mathcal{H}$ with domains $D(A)$ and $D(B)$. Define the commutator $[A, B]$ as

$$[A, B] = AB - BA \quad \text{ on } D([A, B]) = D(AB) \cap D(BA).$$

Proposition 1.1. (Robertson-Schrödinger uncertainty relation).
If A and B are self-adjoint operators and $\alpha, \beta \in \mathbb{C}$, then

$$||(A - \alpha)u|| \; ||(B - \beta)u|| \geq \frac{1}{2}|\langle [A, B]u, u\rangle| \qquad \text{for all } u \in D([A, B]). \tag{1}$$

Proof. Since we have $[(A - \alpha), (B - \beta)] = [A, B]$ for $\alpha, \beta \in \mathbb{C}$, we may assume $\alpha = \beta = 0$. For $u \in D([A, B])$ we have

$$\begin{aligned} |\langle [A, B]u, u\rangle| &= |\langle ABu, u\rangle - \langle BAu, u\rangle| = |\langle Bu, Au\rangle - \langle Au, Bu\rangle| \\ &= 2|Im\langle Au, Bu\rangle| \leq 2|\langle Au, Bu\rangle| \leq 2||Au|| \; ||Bu||. \end{aligned}$$

□

Since in quantum mechanics one postulates that the space of all possible states in the system is the whole Hilbert space $\mathcal{H}$, in a physical context one would like to have that this uncertainty principle holds on $\mathcal{H}$.
In the following example we consider two of the most important self-adjoint operators in quantum mechanics, namely the momentum operator A up to some factor and the position operator B.
Let $\mathcal{H} = L^2([0, 1])$ and

$$A : D(A) \to L^2([0, 1]), Af = if'$$

with $D(A) = \{f \mid f \text{ absolutely continuous on } [0, 1], f' \in L^2([0, 1]), f(0) = f(1)\}$ and

$$B : D(B) \to L^2([0, 1]), Bf : x \mapsto xf(x)$$

with $D(B) = L^2([0, 1])$.
Note that the operators A and B are self-adjoint. Furthermore for $f \in D([A, B]) = \{f | f \text{absolutely continuous on} [0, 1], f' \in L^2([0, 1]), f(0) = f(1) = 0\}$ we have

$$([A, B]f)(x) = i(xf(x))' - x(if'(x)) = ixf'(x) + if(x) - ixf'(x) = if(x).$$

So we have $[A, B] = i\mathcal{I}$ on $D([A, B])$. By definition of the closure of an operator we have $D(\overline{[A, B]}) = \{f \in L^2([0, 1]) | \exists (f_n) \subset D([A, B]) : f_n \to f, [A, B]f_n \text{ converges}\}$. Since $D([A, B])$ is dense in $L^2([0, 1])$ and $[A, B]f_n = if_n \to if$ we have $D(\overline{[A, B]}) = L^2([0, 1])$.

Again by definition of the closure of an operator we have $\overline{[A,B]}f = \lim_{n\to\infty}[A,B]f_n = if$ for some $(f_n) \subset D([A,B])$ such that $f_n \to f$. Thus we have $\overline{[A,B]} = i\mathcal{I}$ on $L^2([0,1])$.
We would like to have

$$\|Af\|\;\|Bf\| \geq \frac{1}{2}|\langle\overline{[A,B]}f,f\rangle| \qquad \text{for all } f \in D(A)\cap D(B)\cap D(\overline{[A,B]}). \tag{2}$$

Since we have $D(A)\cap D(B)\cap D(\overline{[A,B]}) = D(A)$ this would imply for all $f \in D(A)$

$$\|Af\|^2\;\|Bf\|^2 \geq \frac{1}{4}|\langle\overline{[A,B]}f,f\rangle|^2$$

$$\Rightarrow \int_0^1 |if'(x)|^2dx \int_0^1 |xf(x)|^2dx = \|f'\|^2 \int_0^1 x^2|f(x)|^2dx \overset{Plancherel}{=} \|\widehat{f'}\|^2 \int_0^1 x^2|f(x)|^2dx$$

$$\geq \frac{1}{4}|\langle if,f\rangle|^2 = \frac{\|f\|^4}{4}$$

$$\overset{\widehat{f'}(\xi)=2\pi i\xi\widehat{f}(\xi)}{\Longleftrightarrow} \int_0^1 \xi^2|\widehat{f}(\xi)|^2d\xi \int_0^1 x^2|f(x)|^2dx \geq \frac{\|f\|^4}{16\pi^2}.$$

Due to denseness of $D(A)$ and continuity of the scalar product we obtain

$$\int_0^1 \xi^2|\widehat{f}(\xi)|^2d\xi \int_0^1 x^2|f(x)|^2dx \geq \frac{\|f\|^4}{16\pi^2} \quad \text{for all } f \in L^2([0,1])\ .$$

This is Heisenberg's uncertainty relation on $L^2([0,1])$. The quantum mechanical interpretation of this is that we have the uncertainty relation for all states of the system and not just for the subset $D([A,B])$ as in the Robertson-Schrödinger uncertainty relation.
Now we consider the constant function $g \equiv 1$ ($g \in D(A)\cap D(B)\cap D(\overline{[A,B]}) = D(A)$). We have $Ag = 0$ and $|\langle\overline{[A,B]}g,g\rangle| = 1$ in contradiction to (2).

Note that we have $D(A)\cap D(B) \supset D([A,B])$ but in general $D(A)\cap D(B) \not\supset D(\overline{[A,B]})$ and therefore we need $D(A)\cap D(B)\cap D(\overline{[A,B]})$ as domain in (2).
Recall that for Banach spaces X,Y an operator $A: D(A)\to Y,\ D(A)\subset X$ is called closed iff

$$(x_n)_n \subset D(A),\ x_n \to x,\ Ax_n \to y \text{ implies } x \in D(A) \text{ and } Ax = y.$$

If the operator $[A,B]$ is closed then $D(\overline{[A,B]}) = D([A,B])$ and thus in this case (1) and (2) coincide. Since (2) does not hold in the above example, the operator $[A,B]$ is not closed.
Nevertheless it is possible to show the following n-dimensional form of Heisenberg's inequality. For a proof refer to [F].

Theorem 1.2. *Let $f \in L^2(\mathbb{R}^n), a,b \in \mathbb{R}^n$. Then we have*

$$\int \|x-a\|^2|f(x)|^2dx \int \|\xi-b\|^2|\widehat{f}(\xi)|^2d\xi \geq \frac{n^2}{16\pi^2}\|f\|_2{}^4. \tag{3}$$

2 Logarithmic uncertainty inequalities

In this section we deduce a logarithmic uncertainty inequality which implies a sharp form of Heisenberg's n-dimensional uncertainty inequality .
At first we fix some definitions and notations. If a probability measure μ on $\mathbb{R}^n$ has finite variance, i.e. $\int \|x\|^2 d\mu < \infty$, then the mean $M(\mu) \in \mathbb{R}^n$ and the covariance matrix $V(\mu) \in R^{n\times n}$ are defined by

$$M(\mu) = \int x d\mu(x) \text{ and } V(\mu)_{ij} = \int (x - M(\mu))_i (x - M(\mu))_j d\mu(x).$$

We call $\rho \in L^1(\mathbb{R}^n)$ a probability density function if there is a probability measure μ on $\mathbb{R}^n$ such that $d\mu(x) = \rho(x)dx$. In this case we write $M(\rho)$ and $V(\rho)$ instead of $M(\mu)$ and $V(\mu)$.

Definition 2.1. *Let ρ be a probability density function on $\mathbb{R}^n$. We define the entropy of ρ by*

$$E(\rho) = -\int \rho(x) ln\rho(x)dx.$$

Theorem 2.2. *If ρ is a probability density function on $\mathbb{R}^n$ with finite variance, then $E(\rho)$ is well-defined and*

$$E(\rho) \leq \frac{1}{2} ln[(2\pi e)^n detV(\rho)]. \tag{4}$$

Proof. One can show that composing ρ with a translation and a rotation does not affect the quantities in (4) (see appendix 1).
For the translated density function $\rho'(x) = \rho(x - t)$ we have $M(\rho') = M(\rho) + t$. Since translations of ρ do not affect (4), by choosing $t = -M(\rho)$ we can assume that $M(\rho) = 0$.
Since $V(\rho)$ is symmetric and real there is a rotation $R \in \mathbb{R}^{n\times n}$ such that $R^T V(\rho) R$ is diagonal. For the rotated density function $\rho'(x) = \rho(Rx)$ we have $V(\rho') = R^T V(\rho) R$ (see appendix 1). Since rotations of ρ do not affect (4) we can assume that $V(\rho)$ is diagonal.
If we replace ρ by ρ' with $\rho'(x) = \prod_{i=1}^{n} c_i \ \rho(c_1 x_1, ..., c_n x_n)$, $c_i > 0$ for all i, then both sides of (4) decrease by $\sum_i lnc_i$ (see appendix 2). Furthermore we have $V(\rho')_{ii} = \int {x_i}^2 \rho'(x)dx = \frac{1}{{c_i}^2} V(\rho)_{ii}$ and thus by choosing $c_i = \sqrt{V(\rho)_{ii}}$ we can assume that $V(\rho) = \mathcal{I}$.
Now let $\Phi(x) = (2\pi)^{\frac{n}{2}} e^{\frac{\|x\|^2}{2}} \rho(x)$ and $d\gamma(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{\|x\|^2}{2}} dx$, then we obtain $\int \Phi(x) d\gamma(x) = \int \rho(x)dx = 1$.
Since γ is a probability measure and $t \mapsto t\ lnt$ is a convex function, by Jensen's inequality we have

$$\begin{aligned} 0 = \Big[\int \Phi(x)d\gamma(x)\Big] ln\Big[\int \Phi(x)d\gamma(x)\Big] &\leq \int \Phi(x) ln\Phi(x) d\gamma(x) \\ &= \int \rho(x)\big[\frac{n}{2} ln2\pi + \frac{1}{2}\|x\|^2 + ln\rho(x)\big]dx \\ &= \frac{n}{2} ln2\pi \int \rho(x)dx + \frac{1}{2}\int \sum_i {x_i}^2 \rho(x)dx - E(\rho) \\ &= \frac{n}{2} ln2\pi + \frac{1}{2}\sum_i V(\rho)_{ii} - E(\rho). \end{aligned}$$

Thus we have $E(\rho) \leq \frac{n}{2} ln2\pi + \frac{1}{2}\sum_i V(\rho)_{ii}$. Since $V(\rho) = \mathcal{I}$, we have $V(\rho)_{ii} = 1$ and $detV(\rho) = 1$ and therefore we obtain

$$E(\rho) \leq \frac{1}{2} ln[(2\pi e)^n detV(\rho)]. \qquad \square$$

We will make use of the following trivial lemma.

Lemma 2.3. *Suppose $\Phi(t) \leq \Psi(t)$ for $t \in [a,b]$ and $\Phi(a) = \Psi(a)$. If Φ and Ψ are differentiable at a, then $\Phi'(a) \leq \Psi'(a)$.*

We show the central uncertainty inequality in terms of entropy.

Theorem 2.4. *If $f \in L^2(\mathbb{R}^n)$ and $\|f\|_2 = 1$, we have*

$$E(|f|^2) + E(|\widehat{f}|^2) \geq n(1 - ln2). \tag{5}$$

Proof. We consider the sharp Hausdorff-Young inequality

$$\|\widehat{f}\|_q \leq p^{\frac{n}{2p}} q^{-\frac{n}{2q}} \|f\|_p \qquad (1 \leq p \leq 2,\ \frac{1}{p} + \frac{1}{q} = 1).$$

We choose $p = \frac{q}{q-1}$, take both sides of this inequality to the power of q and write it in integral form:

$$\int |\widehat{f}(\omega)|^q d\omega \leq \frac{q}{q-1}^{\frac{n(q-1)}{2}} q^{-\frac{n}{2}} \Big(\int |f(x)|^{\frac{q}{q-1}} dx \Big)^{q-1}. \tag{6}$$

Now we want to apply Lemma 2.3. We consider both sides of the inequality (6) as functions of q, with $q \in [2, 2+\epsilon]$ for some $\epsilon > 0$. We assume that the integrals on both sides of (6) are finite for $q \in [2, 2+\epsilon]$.
For $q = 2$ equality holds in (6).
Differentiating the left-hand side of (6) yields

$$\frac{d}{dq} \int |\widehat{f}(\omega)|^q d\omega = \int |\widehat{f}(\omega)|^q ln|\widehat{f}(\omega)| d\omega$$

and for $q = 2$ we obtain

$$\int |\widehat{f}(\omega)|^2 ln|\widehat{f}(\omega)| d\omega = \frac{1}{2} \int |\widehat{f}(\omega)|^2 ln|\widehat{f}(\omega)|^2 d\omega = -\frac{1}{2} E(|\widehat{f}|^2).$$

Differentiating the right-hand side of (6) we obtain for q=2 (see appendix 3)

$$\frac{n}{2}(ln2 - 1) + \frac{1}{2} E(|f|^2).$$

Using Lemma 2.3 we obtain

$$-\frac{1}{2} E(|\widehat{f}|^2) \leq \frac{n}{2}(ln2 - 1) + \frac{1}{2} E(|f|^2)$$

and thus

$$E(|f|^2) + E(|\widehat{f}|^2) \geq n(1 - ln2). \qquad \square$$

Corollary 2.5. *If $f \in L^2(\mathbb{R}^n)$ and $\|f\|_2 = 1$, then*

$$detV(|f|^2)\ detV(|\widehat{f}|^2) \geq (16\pi^2)^{-n}. \tag{7}$$

Proof. We have

$$ln\Big[(2\pi e)^{2n} detV(|f|^2)\ detV(|\widehat{f}|^2)\Big] = ln\Big[(2\pi e)^n detV(|f|^2)\Big] + ln\Big[(2\pi e)^n detV(|\widehat{f}|^2)\Big]$$

$$\overset{(4)}{\geq} 2(E(|f|^2) + E(|\widehat{f}|^2)) \overset{(5)}{\geq} 2n(1 - ln2) \quad \text{and thus}$$

$$detV(|f|^2)\ detV(|\widehat{f}|^2) \geq (2\pi e)^{-2n} e^{2n-2nln2} = (4\pi^2)^{-n} 2^{-2n} = (16\pi^2)^{-n}. \qquad \square$$

Since we can assume that $V(\rho)$ is diagonal for any probability density function ρ, by the inequality of arithmetic and geometric means we obtain

$$[detV(\rho)]^{\frac{1}{n}} = \Big(\prod_{i=1}^{n} V(\rho)_{ii}\Big)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} V(\rho)_{ii} = \frac{1}{n}\int \sum_{i=1}^{n} (x - M(\rho))_i{}^2 \rho(x)dx$$
$$= \frac{1}{n}\int \|x - M(\rho)\|^2 \rho(x)dx.$$

Thus Corollary 2.5 is a sharp form of the n-dimensional Heisenberg inequality (3).

Appendix

1. Invariance of the quantities in equation (4) under translations and rotations of the density function ρ

1. Composing ρ with a translation does not affect $E(\rho)$.
Consider $\rho \mapsto \rho',\ \rho'(x) = \rho(x-t)$. We have

$$E(\rho') = -\int \rho'(x) ln\rho'(x) dx = -\int \rho(\underbrace{x-t}_{:=x'}) ln\rho(x-t) dx$$
$$= -\int \rho(x') ln\rho(x') \underbrace{|detJ|}_{=1} dx' = E(\rho),$$

since $x = x' + t$ => $\frac{\partial x_i}{\partial x'_k} = \delta_{ik}$ => Jacobian matrix $J = \mathcal{I}$ => $detJ = 1$.

2. Composing ρ with a translation does not affect $V(\rho)$.
Consider $\rho \mapsto \rho',\ \rho'(x) = \rho(x-t)$. Using

$$M(\rho') = \int x\rho'(x)dx = \int x\rho(\underbrace{x-t}_{:=x'})dx = \int (x'+t)\rho(x') \underbrace{|detJ|}_{=1} dx'$$
$$= M(\rho) + t \underbrace{\int \rho(x')dx'}_{=1} = M(\rho) + t$$

we obtain

$$V(\rho')_{ij} = \int (x - M(\rho'))_i (x - M(\rho'))_j \rho'(x) dx$$
$$= \int (x - M(\rho) - t)_i (x - M(\rho) - t)_j \rho(\underbrace{x-t}_{:=x'}) dx$$
$$= \int (x' - M(\rho))_i (x' - M(\rho))_j \rho(x') \underbrace{|detJ|}_{=1} dx' = V(\rho)_{ij}.$$

3. Composing ρ with a rotation does not affect $E(\rho)$.
A rotation is a map $R : \mathbb{R}^n \to \mathbb{R}^n$ with $R^T R = \mathcal{I}, RR^T = \mathcal{I}$, i.e. $R^T = R^{-1}$.
Consider $\rho \mapsto \rho',\ \rho'(x) = \rho(Rx)$. We have

$$E(\rho') = -\int \rho'(x) ln\rho'(x) dx = -\int \rho(\underbrace{Rx}_{=:x'}) ln\rho(Rx) dx$$
$$= -\int \rho(x') ln\rho(x') \underbrace{|detJ|}_{=1} dx' = E(\rho),$$

since $x = R^{-1}x'$ => $x_i = \sum_j R^{-1}_{ij} x'_j$ => $\frac{\partial x_i}{\partial x'_k} = R^{-1}_{ik}$ => Jacobian matrix $J = R^{-1}$ and we have $1 = det(R^T R) = [det(R^T)]^2 = [det(R^{-1})]^2 = [det(J)]^2)$ => $|detJ| = 1$.

4. Composing ρ with a rotation does not affect $detV(\rho)$.
Consider $\rho \mapsto \rho',\ \rho'(x) = \rho(Rx)$. We have

$$M(\rho') = \int x\rho'(x)dx = \int x\rho(\underbrace{Rx}_{:=x'})dx = \int R^{-1}x'\rho(x')\underbrace{|detJ|}_{=1}dx' = R^{-1}M(\rho)$$

and therefore

$$\begin{aligned} V(\rho')_{ij} &= \int (x - M(\rho'))_i(x - M(\rho'))_j\rho(\underbrace{Rx}_{:=x'})dx \\ &= \int [R^{-1}(x' - M(\rho))]_i[R^{-1}(x' - M(\rho))]_j\rho(x')\underbrace{|detJ|}_{=1}dx' \\ &= \int \sum_{k,l} R^{-1}_{ik}(x' - M(\rho))_k R^{-1}_{jl}(x' - M(\rho))_l\rho(x')dx' \\ &= \sum_{k,l} R^{-1}_{ik}R^{-1}_{jl}\int (x' - M(\rho))_k(x' - M(\rho))_l\rho(x')dx' \\ &= \sum_{k,l} R^T_{ik}V(\rho)_{kl}R_{lj} \\ &= (R^TV(\rho)R)_{ij}. \end{aligned}$$

Thus we have $V(\rho') = R^TV(\rho)R$ and therefore
$detV(\rho') = det(R^TV(\rho)R) = det(R^TR)detV(\rho) = detV(\rho)$.

2. Effects of the transformation $\rho(x) = \prod_{i=1}^{n} c_i\rho(c_1x_1, ..., c_nx_n)$ on equation (4)

Replace ρ by $\prod_{i=1}^{n} c_i\rho(c_1x_1, ..., c_nx_n) = detC\rho(Cx)$, for $C \in \mathbb{R}^{n\times n}$ with $C_{ij} = c_i\delta_{ij}$.
We show that by this transformation both sides of (4) decrease by $\sum_i lnc_i$.
Let $\rho \mapsto \rho', \rho'(x) = detC\rho(Cx)$.
For the left side of (4) we have

$$\begin{aligned} E(\rho') &= -\int detC\rho(\underbrace{Cx}_{:=x'})ln[detC\rho(Cx)]dx \\ &= -detC\int \rho(x')[ln(detC) + ln(\rho(x'))]\underbrace{|detJ|}_{=detC^{-1}}dx' \\ &= E(\rho) - ln\prod_i c_i = E(\rho) - \sum_i lnc_i, \end{aligned}$$

since $x = C^{-1}x'$ with $C^{-1}_{ij} = \frac{1}{c_i}\delta_{ij} \Rightarrow x_i = \frac{1}{c_i}x'_i \Rightarrow \frac{\partial x_i}{\partial x'_k} = \frac{1}{c_i}\delta_{ik} \Rightarrow$ Jacobian matrix $J = C^{-1}$.
For the left side we first notice that

$$V(\rho')_{ii} = \int {x_i}^2 detC\rho(\underbrace{Cx}_{:=x'})dx = detC\int \frac{1}{c_i^2}{x'_i}^2\rho(x')detC^{-1}dx' = \frac{1}{c_i^2}V(\rho)_{ii}.$$

Thus we obtain

$$\begin{aligned}\frac{1}{2}ln[(2\pi e)^n detV(\rho')] &= \frac{1}{2}ln[(2\pi e)^n \prod_i V(\rho')_{ii}] \\ &= \frac{1}{2}ln[(2\pi e)^n \prod_i \frac{1}{c_i^2}V(\rho)_{ii}] \\ &= \frac{1}{2}ln[(2\pi e)^n detV(\rho)] + \frac{1}{2}ln\prod_i \frac{1}{c_i^2} \\ &= \frac{1}{2}ln[(2\pi e)^n detV(\rho)] - \sum_i lnc_i.\end{aligned}$$

3. Differentiating the right hand side of the Hausdorff-Young inequality

The right hand side of the Hausdorff-Young inequality (to the power of q) in integral form is

$$f(q) := \underbrace{\frac{q}{q-1}^{\frac{n(q-1)}{2}}}_{:=a(q)} \underbrace{q^{-\frac{n}{2}}}_{:=b(q)} \underbrace{\Big(\int |f(x)|^{\frac{q}{q-1}}dx\Big)^{q-1}}_{:=c(q)}.$$

Note that $||f||_2 = 1$ by assumption in the Theorem.
We have

$$a'(q) = \frac{q}{q-1}^{\frac{n(q-1)}{2}}\Big[-\frac{n}{2q} + \frac{n}{2}ln(\frac{q}{q-1})\Big] \;\; => \;\; a'(2) = 2^{\frac{n}{2}}\frac{n}{4}(-1+ln2) + n\ ln2\ 2^{\frac{n}{2}-2}$$

$$b'(q) = -\frac{n}{2}q^{-\frac{n+2}{2}} \;\; => \;\; b'(2) = -\frac{n}{2}2^{-\frac{n+2}{2}}$$

and

$$\begin{aligned}c'(q) = \Big(\int |f(x)|^{\frac{q}{q-1}}dx\Big)^{q-1}\Big(&\frac{1}{\int |f(x)|^{\frac{q}{q-1}}dx}\Big[\int |f(x)|^{\frac{q}{q-1}}ln|f(x)|\frac{-1}{(q-1)^2}dx\Big](q-1) \\ &+ ln\Big(\int |f(x)|^{\frac{q}{q-1}}dx\Big)\Big)\end{aligned}$$

$$=> \;\; c'(2) = \frac{1}{2}E(|f|^2)$$

Thus we have

$$\begin{aligned}f'(2) &= a'(2)b(2)c(2) + a(2)b'(2)c(2) + a(2)b(2)c'(2) \\ &= (2^{\frac{n}{2}}\frac{n}{4}(-1+ln2) + n\ ln2\ 2^{\frac{n}{2}-2})\ 2^{-\frac{n}{2}}\ ||f||_2{}^2 \\ &+ 2^{\frac{n}{2}}\ (-\frac{n}{2}\ 2^{-\frac{n+2}{2}})\ ||f||_2{}^2 \\ &+ 2^{\frac{n}{2}}\ 2^{-\frac{n}{2}}\ \frac{1}{2}E(|f|^2) \\ &= \frac{n}{2}(ln2-1) + \frac{1}{2}E(|f|^2).\end{aligned}$$

Reference

[F] Gerald B. Folland, Alladi Sitaram. The Uncertainty Principle: A Mathematical Survey, The Journal of Fourier Analysis and Applications, 1997.