Hauptseminar Uncertainty Relation

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„Uncertainty Relation in Signal Recovery"

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1 Introduction

The uncertainty principle is mostly used to show that something is not possible. For example determining the momentum and position of a particle simultaneously or measuring the "instantaneous (momentane) frequency" of a signal. The paper from D. L. Donoho and P. B. Stark point out that something unexpected is possible, with the help of the generalized uncertainty principle - the recovery of a signal or image despite significant amounts of missing information. Before using the generalized uncertainty relation for recovering missing segments of a bandlimited signal, we need to define some restrictions (i.e. bandlimited) and mathematical relations, such as Fourier Transform.

1.1 Fourier Transformation

Let $f(t)$ be a complex-valued function of $t \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{C}$. The corresponding *fourier transform* changes from the time domain to frequency domain (and back - *inverse fourier transform*) by

$$\hat{f}(w) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iwt} dt.$$

For example we have as fourier transform of $f(t)$ the function $\hat{f}(\omega) = \dfrac{1}{1 + c \cdot \omega^8}$, with c constant.

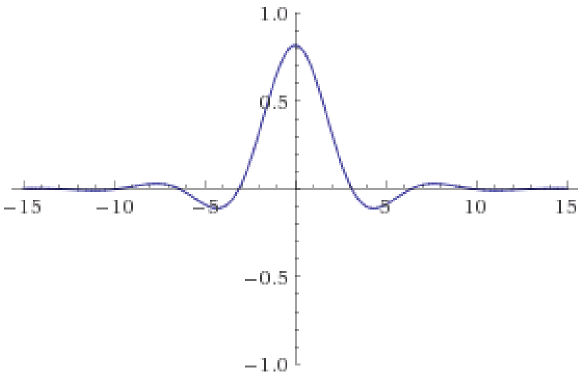

Figure 1: $f(t)$ with $c = 1$

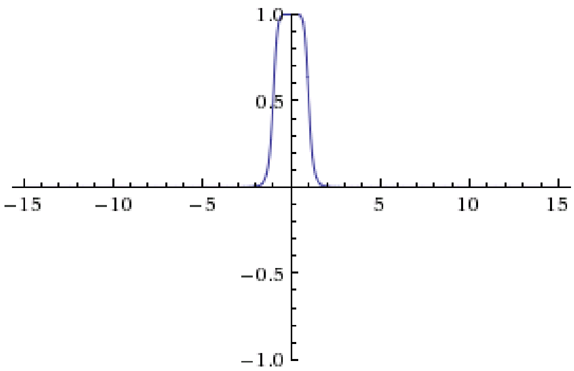

Figure 2: $\hat{f}(\omega) = \dfrac{1}{1 + \omega^8}$

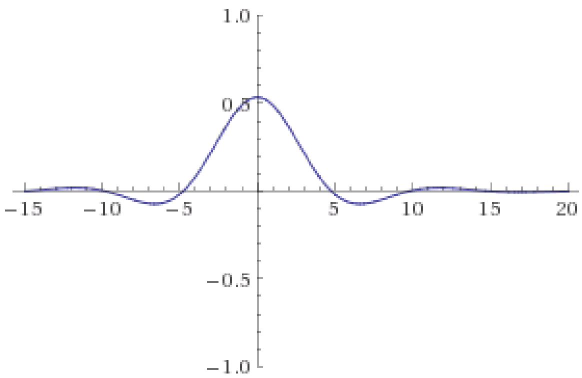

Figure 3: $f(t)$ with $c = 30$

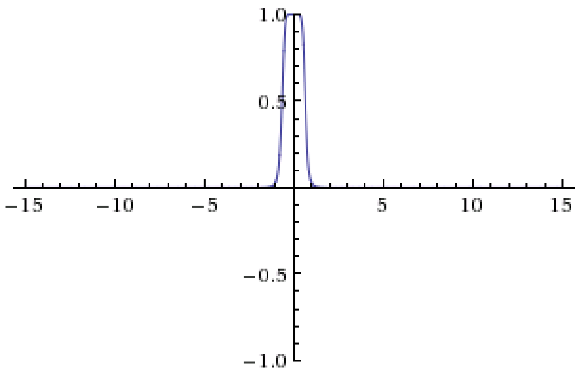

Figure 4: $\hat{f}(\omega) = \dfrac{1}{1 + 30 \cdot \omega^8}$

In the pictures above, we see the change from time domain to frequency domain of the given functions. In the upper graphics is the constant $c = 1$ where in the lower graphics the constant was set to 30, so that we can see how f and $\hat{f}$ are related to each other. If f got squeezed from above then $\hat{f}$ becomes more narrow, but still has it's maximum at 1.

The L^2-norm of f is defined as follows

$$\|f\| \equiv \sqrt{\int_{-\infty}^{\infty} \mid f(t) \mid^2 dt}.$$

And with the Parseval's identity the norm of $\hat{f}$ is

$$\text{(Parseval's identity)} \qquad \int \mid f(t) \mid^2 = \int \mid \hat{f}(w) \mid^2.$$

In the paper of D. L. Donoho and P. B. Stark they suppose that the norm of both f and $\hat{f}$ is one. But in fact we needn't care about that, because we can just rescale the result of the norm with any constant so that it becomes one.

1.2 ε-concentration

The general uncertainty principle says that f and its fourier transform $\hat{f}$ cannot be both highly concentrated, that means that the more information is known about t, the less is known about w. But in our case for signal recovery we can use a more general version of the uncertainty principle in which f and $\hat{f}$ needn't be concentrated on an interval. Say that $\mid T \mid$ and $\mid W \mid$ are measures of the sets T and W. This means

$$T \in B(\mathbb{R}) : \mid T \mid = \int_T dt$$
$$W \in B(\mathbb{R}) : \mid W \mid = \int_W dt,$$

where $B(\mathbb{R})$ are the Borel sets on $\mathbb{R}$ and $\int dt$ the Lebesue measure.
f is said to be ε_T*-concentrated* on a measurable set T if $\exists g(t)$, which vanishes outside T, s.t.

$$\|f - g\| \leq \varepsilon_T \|f\|.$$

The same definition holds for $\hat{f}$ w.r.t. a measurable set W. So we say $\hat{f}$ is ε_W*-concentrated* if $\exists h(w)$, which vanishes outside W, s.t.

$$\|\hat{f} - h\| \leq \varepsilon_W \|\hat{f}\|.$$

2 L^2 continous-time uncertainty Relation

2.1 Time-limiting and frequency-limiting Operator

For $T, W \in B(\mathbb{R})$ let $P_T, P_W : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ be the time-limiting and frequency-limiting operators defined as follows.

Time-limiting Operator

$$(P_T f)(t) \equiv \begin{cases} f(t), & t \in T \\ 0, & otherwise \end{cases}$$

This Operator sets the part of the function f outside the interval T to zero. That means it gives the closest function to f (in the L^2-norm) that vanishes off T. Thus f is ε_T-concentrated on T iff $\|f - P_T f\| \leq \varepsilon_T \|f\|$.

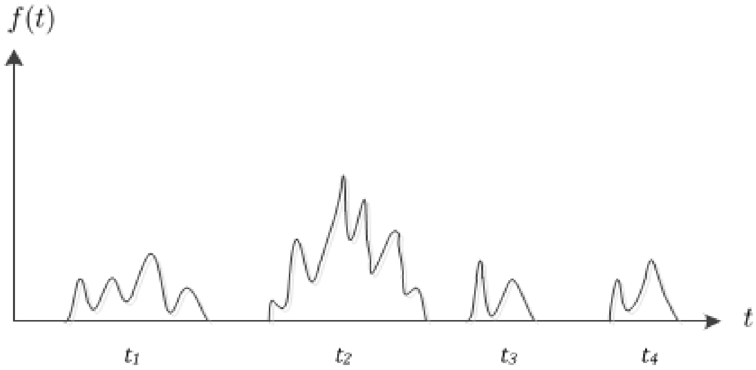

Figure 5: Time-limiting Operator

In the figure above we see an example for time-limiting, here is $T = [t_i | i = 1, ..., 4]$. The signal is zero outside the measurable set T.

Frequency-limiting Operator

$$(P_W f)(t) \equiv \int_W e^{2\pi i w t} \hat{f}(w) dw$$

$(P_W f)$ is a partial reconstruction of f, which uses only frequency information inside the interval W. If $g = (P_W f)$ then $\hat{g}$ vanishes outside W. Thus $\hat{f}$ is ε_W-concentrated on W iff $\|f - P_W f\| \leq \varepsilon_W \|f\|$. Note here that Parseval's identity holds.

An example for bandlimited is, that a loud speaker generates only frequenciess between 20Hz and 20.000Hz. This is because humans can hear only frequencies in the interval $W = [20\text{Hz}, 20.000\text{Hz}]$. So the band-limiting Operator $P_W f$ cuts all frequencies outside the range of W.

2.2 L^2 uncertainty relation

Theorem 1

Let $T, W \in B(\mathbb{R})$ and suppose there is a fourier transform pair $(f, \hat{f})$, with f and $\hat{f}$ of unit norm, such that f is ε_T-concentrated on T and $\hat{f}$ is ε_W-concentrated on W. Then

$$| T | \cdot | W | \geq \|P_W P_T\|^2 \geq (1 - (\varepsilon_T + \varepsilon_W))^2.$$

(Where the second inequality is the sharper one and the fundamental statement w.r.t uncertainty relations in signal recovery.)

<u>Proof</u>

First inequality: $| T | \cdot | W | \geq \|P_W P_T\|^2$

$$\begin{aligned} P_W P_T f(s) &= \int_W e^{2\pi i \omega s} \int_T e^{-2\pi i \omega t} f(t) dt d\omega \\ &= \int_T \int_W e^{2\pi i (s-t)\omega} d\omega f(t) dt \\ &= \int_{\mathbb{R}} ker(s,t) f(t) dt \Longrightarrow P_W P_T \text{ is compact operator} \end{aligned}$$

$$\Longrightarrow \|P_W P_T\|^2 \leq \|P_W P_T\|_2^2 = \textstyle\int_T \int_W d\omega dt = | T || W |$$

□

Second inequality: $\|P_W P_T\|^2 \geq (1 - (\varepsilon_T + \varepsilon_W))^2$

$$\begin{aligned} \|f - P_W P_T f\| &= \|f - P_W f + P_W f - P_W P_T f\| \\ &= \|f - P_W f + P_W (f - P_T f)\| \\ &\leq \|f - P_W f\| + \|P_W (f - P_T f)\| \\ &\leq \|f - P_W f\| + \|P_W\| \|(f - P_T f)\| \\ &\leq (\varepsilon_W + \varepsilon_T) \|f\| \end{aligned}$$

$$\begin{aligned} \|f - P_W P_T f\| &\geq \|f\| - \|P_W P_T f\| \\ (\varepsilon_W + \varepsilon_T)\|f\| &\geq \|f\| - \|P_W P_T f\| \\ (\varepsilon_W + \varepsilon_T) &\geq 1 - \frac{\|P_W P_T f\|}{\|f\|} \\ \frac{\|P_W P_T f\|}{\|f\|} &\geq 1 - (\varepsilon_W + \varepsilon_T) \\ \|P_W P_T\| &\geq 1 - (\varepsilon_W + \varepsilon_T) \end{aligned}$$

□

3 Uncertainty Relation in Signal Recovery

3.1 Signal Recovery

Let $s(t) \in L^2$ be a signal, which is bandlimited. In the previous section we used P_W as the bandlimiting Operator, this means for this section that

$$P_W s = s.$$

In other words, the bandlimiting operator applied to the signal just "cuts" the rest of the signal we needn't care about; for example the human hearing in the previous section. So the signal is equal to the signal which is already bandlimited. Now we consider the receiver, which knows that s is bandlimited and cannot observe all of the signal s. Let's say that the signal in a certain subset T^C can't be received correctly due of noise $n(t) \in L^2$ and for the other $t \in T$ we don't have any signal, not even any information about it. This means for the received signal

$$r(t) = \begin{cases} s(t) + n(t), & t \in T^C \\ 0, & t \in T \end{cases}$$

where T^C is the complement of the set T and wlog we assume that $n = 0$ for signals $s(t), t \in T$. Equivalently we can write r as follows

$$r = (id - P_T)(s + n)$$

By the uncertainty principle the reconstruction of s from the noisy received signal r is possible provided $| T || W |< 1$.

Theorem 2

If $| T || W |< 1$ holds, then there is a recovery operator $Q : L^2 \to L^2$ s.t.

$$\|s - Qr\| \le \frac{\|n\|}{1 - \|P_W P_T\|} \le \frac{\|n\|}{1 - \sqrt{| T || W |}}$$

Proof

Let $Q = \dfrac{1}{id - P_T P_W} = \sum_{k=0}^{\infty} (P_T P_W)^k$.

Q exits because of $\|P_T P_W\| = \|P_W P_T\| \le \sqrt{| T || W |} < 1$. We also use in the proof that

$$\|Q\| = \|\frac{1}{id - P_T P_W}\| \le \frac{1}{1 - \|P_T P_W\|}.$$

$$\begin{aligned}
\|s - Qr\| &= \|s - Q(id - P_T)(s + n)\| \\
&= \|s - Q(id - P_T)s - Q(id - P_T)n\| \\
&= \|s - Qids + QP_T s - Q(id - P_T)n\| \\
&= \|s - Qids + QP_T P_W s - Q(id - P_T)n\| \\
&= \|s - Q(id - P_T P_W)s - Q(id - P_T)n\| \\
&= \|s - s - Q(id - P_T)n\| \\
&= \| - Q(id - P_T)n\| \\
&= | -1 | \, \|Q(id - P_T)n\| \\
&= \|Q(id - P_T)n\| \\
&\le \|Q\|\|id - P_T\|\|n\| \\
&= \|Q\|\|n\| \\
&\le \frac{\|n\|}{1 - \|P_W P_T\|}
\end{aligned}$$

□

3.2 Alternating Projection Method

In the proof for the previous theorem we used the Neumann Series

$$Q = \frac{1}{id - P_T P_W} = \sum_{k=0}^{\infty} (P_T P_W)^k.$$

We then can compute Qr by writing $s^{(n)}$ as follows

$$s^{(n)} = \sum_{k=0}^{\infty} (P_T P_W)^k r$$

and iterating $s^{(n)}$ for $n \to \infty$ with the following algorithm

$$\begin{aligned} s^{(0)} &= r \\ s^{(1)} &= r + P_T P_W s^{(0)} \\ s^{(2)} &= r + P_T P_W s^{(1)} \\ &\dots \end{aligned}$$

The goal here is to reconstruct the missing segments of the signal. The iterate $s^{(n)}$ is the result of bandlimiting then timelimiting $s^{(n-1)}$, then adding the result back to the original data r. The iterations converge at a geometric rate to the fixed point

$$s^* = r + P_T P_W s^*.$$

On T^c where the data is observed, $s^{(n)} = r$ at each iteration n, while on the unobserved set T the missing values are filled in by a gradual adjustment.
By reforming $s^* = r + P_T P_W s^*$ as follows

$$\begin{aligned} s^* &= r + P_T P_W s^* \\ s^* - P_T P_W s^* &= r \\ s^*(id - P_T P_W) &= r \\ s^* &= \frac{r}{id - P_T P_W} \\ \implies s^* &= Qr \end{aligned}$$

we see that the equation satisfies *Theorem 2* of the previous section.
The algorithm is an instance of the *alternating projection method*: it alternately applies the bandlimiting projector P_W and the timelimiting projector P_T.

Quellenverzeichnis

[1] David L. Donoho, Philip B. Stark
"Uncertainty Relations and Signal Recovery"
Kapitel 1 "Introduction"
Kapitel 3 "The continous-time principle"
Kapitel 4 "Recovering missing segments of a bandlimited signal"

[2] Michael M. Wolf
Lecture WS 12/13 "Information Theory"
Kapitel "Signal Recovery & Uncertainty Relations"