Uncertainty Relations for Quantum Mechanical Observables

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Introduction

Probably the most famous result from quantum theory and the best-known uncertainty relation is Heisenberg's uncertainty relation. Yet its result is often interpreted in a wrong way.

1 Definitions

We start with a basic definition of quantum mechanics, that fits our purposes (for more details, see [1]). We set $(H, \langle\, .\, |\, .\, \rangle)$ to be a Hilbert space. Usually we have $(H, \langle\, .\, |\, .\, \rangle) = (L^2(\mathbb{R}), \langle\, .\, |\, .\, \rangle_2)$, with

$$\langle f|g\rangle_2 = \int_{\mathbb{R}} \overline{f(x)} g(x)\, dx \qquad \text{for all } f, g \in H.$$

We call a linear operator $A : \mathscr{D}(A) \to H$, $\mathscr{D}(A)$ being a linear subset of H, self-adjoint iff $\langle A\psi|\phi\rangle = \langle\psi|A\phi\rangle$ for all ϕ, ψ in $\mathscr{D}(H)$. The *domain* $\mathscr{D}(A)$ of a linear operator A can always supposed to be the maximal linear subspace of H, on which A is defined.

- A quantummechanical *state* is a vector $\psi \in H$ with normalization $\|\psi\| = 1$.
- An *observable* is a linear, self-adjoint operator $A : \mathscr{D}(A) \to (H)$.
- The *possible values* of an observable are exactly its eigenvalues.
- Suppose we measure an observable A of a state ψ. The *probability* p_λ that the eigenvalue λ is retutned is defined by

  $$p_\lambda = \sum_{i\in I} |\langle\psi|\psi_i\rangle|^2,$$

  with $\{\psi_i\}_{i\in I}$ being a ONB for the eigenspace corresponding to the eigenvalue λ.
- The *average value* of an observable A of the state $\psi \in H$ is defined as $\langle\psi|A\psi\rangle =: \langle A\rangle_\psi$. We will use the notation $\langle\psi|A\psi\rangle = \langle A\rangle_\psi$ for all linear operators A (not only observables).

Remarks

- For an self-adjoint operator all eigenvalues are real, which makes it easier to interpret its possible values as measurement results. Also $\langle A\rangle_\psi$ is real for all $\psi \in \mathscr{D}(A)$.
- We quote a result from spectral theory:
  Let $A \in B(H)$, A normal (i.e. $\|A\psi\| = \|A^*\psi\|$ for all $\psi \in H$) with $\sigma(A) = \{\lambda_1, \lambda_2, \dots\}$ coutable.
  $\sigma(A) := \{\lambda \in \mathbb{C} : (A - \lambda\mathbb{1}) \text{ is not bijective}\}$ is defined as the *spectrum* of A.
  Then there is a ONB $(\psi_i)_{i\in\mathbb{N}}$ of H s.th. $A\psi_i = \lambda_i\psi_i$ for all $i \in \mathbb{N}$. In this case the average value is just the weighted sum of the possible values:

  $$\langle\psi|A\psi\rangle = \langle \sum_{i\in\mathbb{N}} \langle\psi|\psi_i\rangle\psi_i \mid A \sum_{i\in\mathbb{N}} \langle\psi|\psi_i\rangle\psi_i \rangle = \sum_{i\in\mathbb{N}} \overline{\langle\psi|\psi_i\rangle}\langle\psi|\psi_i\rangle\langle\psi_i|A\psi_i\rangle = \sum_{i\in\mathbb{N}} |\langle\psi|\psi_i\rangle|^2\lambda_i$$

  Since $\sum_{i\in\mathbb{N}} |\langle\psi|\psi_i\rangle|^2 = 1$, our definition of average value coincides with the statistical interpretaion of the probability.

- The operator for the observalble *momentum* in $L^2(\mathbb{R})$ is $P : \mathscr{D}(P) \to H,\ \psi \mapsto -i\frac{d}{dx}\psi(x)$, with $\mathscr{D}$ being a dense subset of $L^2(\mathbb{R})$.
- The operator for the opservable *position* in $L^2(\mathbb{R})$ is $Q : \mathscr{D}(Q) \to H,\ \psi(.) \mapsto (.)\psi(.)$, with $\mathscr{D}(Q) = \{\psi \in H \,|\, Q\psi \in H\}$.

2 Heisenberg's position-momentum uncertainty

For operators A, B we set $\{A, B\}_+ = AB + BA$ and $[A, B] = AB - BA$.

Theorem 2.1. *Let A, B be self-adjoint operators, then we have for all states $\psi \in \mathscr{D}(AB) \cap \mathscr{D}(BA)$:*

$$\|A\psi\|\|B\psi\| \geq \frac{1}{2}\left[\langle\{A, B\}_+\rangle_\psi^2 + |\langle[A, B]\rangle_\psi|^2\right]^{1/2}.$$

Equality holds iff $A\psi$ and $B\psi$ are lineary dependent.

Definition 2.2. *Let A, B be self-adoint operators and $\psi \in \mathscr{D}(AB) \cap \mathscr{D}(BA)$ be a state.*

i) We define the uncertainty or standard deviation of A of the state ψ as follows:

$$\Delta_\psi(A) := \left[\|A\psi\|^2 - \langle A\rangle_\psi^2\right]^{1/2}$$

ii) The covariance of A and B is defined as

$$\mathrm{cov}_\psi(A, B) := \frac{1}{2}\langle\{A - a\,\mathrm{id}, B - b\,\mathrm{id}\}_+\rangle_\psi,$$

with $a := \langle A\rangle_\psi$ and $b := \langle B\rangle_\psi$.

Corollary 2.3 (Robertson-Schrödinger)**.** *Let A, B be self-adjoint operators, then we have for all states $\psi \in \mathscr{D}(AB) \cap \mathscr{D}(BA)$:*

$$\Delta_\psi(A)\Delta_\psi(B) \geq \left[\mathrm{cov}_\psi(A, B)^2 + \frac{1}{4}|\langle[A, B]\rangle|^2\right]^{1/2},$$

where equality holds iff $(A - a\,\mathrm{id})\psi$ is a scalar multiple of $(B - b\,\mathrm{id})\psi$. Often the weaker version

$$\Delta_\psi(A)\Delta_\psi(B) \geq \frac{1}{2}|\langle[A, B]\rangle_\psi| \tag{1}$$

is used.

Theorem 2.4 (Heisenbergs position-momentum uncertainty)**.** *For the position operator Q and the momentum operator P we have in any state $\psi \in \mathscr{D}(PQ) \cap \mathscr{D}(QP)$:*

$$\Delta_\psi(Q)\Delta_\psi(P) \geq \frac{1}{2} \tag{2}$$

3 Heisenbergs noise-disturbance uncertainty

Often the position-momentum uncertainty (2) is interpreted in the following setup: The observable A of the particle in the state ψ is measured with error $\eta_\psi(A)$. By this measurement noise $\epsilon_\psi(B)$ is inflicted on the state ψ. This noise is added to the measurement of B, which is therefore imprecise. The statement in this context is then

$$\epsilon_\psi(A)\eta_\psi(B) \geq 1/2|\langle A, B\rangle_\psi|,$$

which in general is false (cf. [2]).
To gain a solid uncertainty, we specify the experiment and especially the measuring process more: Let ψ, ξ be two states (representing particles). We first want to measure A on ψ. We assume that every meaurement includes interaction with another particle (cf. measurement of car speed with radar gun). So for the A-measurement, ψ interacts with ξ. Then a third observable M of ξ is supposed to have information on A of ψ. Also, after interacion, A is measured on ψ.

Quantum theory postulates, that the combined system of ψ and ξ is described by their tensor product $\psi\otimes\xi$ in the Hilbert space $H\otimes H = L^2(\mathbb{R})\otimes L^2(\mathbb{R})$. In this space the the Observables A and B of ψ become $A^{in} := A\otimes \mathrm{id}$ and $B^{in} := B\otimes \mathrm{id}$, respectively, and the observable M of ξ becomes $\mathrm{id}\otimes M$, which are linear, self-adjoint operators on $H\otimes H$.
The interaction is now postulated to be a unitary operator U acting on $H\otimes H$. After interaction the combined system is in the state $U(\psi\otimes\xi)$. Concerning the observables, we get for the measurement after interaction:

$$\langle A^{in}\rangle_{U\psi\otimes\xi} = \langle U\,\psi\otimes\xi|A^{in}U\,\psi\otimes\xi\rangle = \langle\psi\otimes\xi|U^* A^{in}U\,\psi\otimes\xi\rangle = \langle U^* A^{in}U\rangle_{\psi\otimes\xi} =: \langle A^{out}\rangle_{\psi\otimes\xi}$$

Analogously we define $B^{out} = U^* B^{in} U$ and $M^{out} := U^* M^{in} U$, which equal the according observables after interaction. We also introduce the *noise operator* $N(A)$ and the *disturbance operator* $D(B)$ by

$$N(A) := M^{out} - A^{in},$$
$$D(B) := B^{out} - B^{in}.$$

As the difference of self-adjoint operators, they are self-adjoint. For quantification we set the *noise* as

$$\epsilon_{\psi\otimes\xi}(A) := \langle (M^{out} - A^{in})^2\rangle^{1/2}_{\psi\otimes\xi} = \langle N(A)^2\rangle^{1/2}_{\psi\otimes\xi} \geq \Delta_{\psi\otimes\xi}(N(A))$$

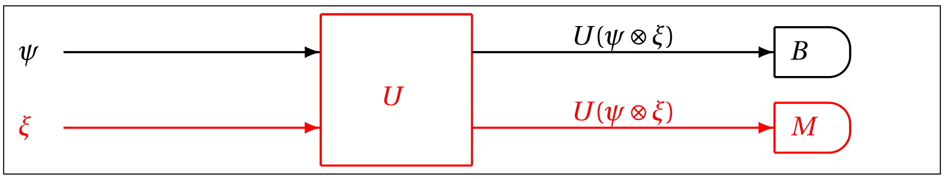

Figure 1: A general scheme for a measurement. The red elements are used to determine the observable A of the state ψ.

and the *disturbance* as

$$\eta_{\psi\otimes\xi}(B) := \langle (B^{out} - B^{in})^2 \rangle^{1/2}_{\psi\otimes\xi} = \langle D(B)^2 \rangle^{1/2}_{\psi\otimes\xi} \geq \Delta_{\psi\otimes\xi}(D(B)).$$

Because of the identity in the definitions of M^{out} and B^{out} they commute: $[M^{out}, B^{out}] = 0$. Then

$$[N(A), D(B)] + [N(A), B^{in}] + [A^{in}, D(B)] = -[A^{in}, B^{in}]$$
$$|\langle [N(A), D(B)] \rangle_{\psi\otimes\xi}| + |\langle [N(A), B^{in}] \rangle_{\psi\otimes\xi}| + |\langle [A^{in}, D(B)] \rangle_{\psi\otimes\xi}| \geq |\langle [A,B] \rangle_{\psi}|.$$

where we used that $\langle [A^{in}, B^{in}] \rangle_{\psi\otimes\xi} = \langle [A,B] \rangle_{\psi}$. With the weaker version of the Robertson inequality (1), we get

$$\epsilon_{\psi}(A)\eta_{\psi}(B) + \epsilon_{\psi}(A)\Delta_{\psi}(B) + \Delta_{\psi}(A)\eta_{\psi}(B) \geq \frac{1}{2}|\langle [A,B] \rangle_{\psi}|.$$

For more information on this section, see ([3]).

References

[1] L. E. Ballentine. The statistical interpretation of quantum mechanics. 42(4):358ff, October 1970.

[2] Jacqueline Erhart Georg Sulyok, Stephan Sponar. Violation of heisenberg's error-disturbance uncertainty relation in neutron spin measurements, May 2013. arXiv:1305.7251v1 [quant-ph].

[3] Masanao Ozawa. Universally valid reformulation of the heisenberg uncertainty principle on noise and disturbance in measurement. *Physical review*, 2003.