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1. Quantum mechanics - Introduction

Before we can turn our attention to time-energy uncertainty relations we need to introduce some concepts of quantum mechanics.
For a measurement of a physical quantity at a macroscopic scale every repeat of an experiment with the exact same initial condition will always result in the same value of that quantity. For a measurement at a microscopic scale this is different. There, the repeat of an experiment given the exact same initial condition for every run will lead to different results. This is not only explainable by an inaccuracy of the measurement device, but rather due to the statistical character of quantum mechanics.
The time-energy uncertainty relations will yield that the variation of a measurement is bounded from below.

Definition 1. (State space & state)
In the general case, a state space is a complex Hilbert space $\mathscr{H}$. At any time a state of a physical system will be identified with an element ψ in the state space $\mathscr{H}$ such that $|\psi| = 1$. We will restrict ourselves in the presentation to the case, where $\mathscr{H} = \mathbb{C}^n$ with standard inner product. The inner product is the sesquilinear form which is anti linear in the first and linear in the second argument. A single state is represented by a column vector $|\psi\rangle := (\psi_1, \cdots, \psi_n)^T \in \mathscr{H}$. The vector in that form is called a ket. We write $\langle\phi|$ for the linear operator such that $\langle\phi| : |\psi\rangle \mapsto \langle\phi|\psi\rangle$ and call it bra.

An example for a state vector would be the two dimensional vector that describes the spin of an electron.

In our view, the restriction to the finite dimensional case is legitimate, since this already covers several interesting physical situations and suffices to illustrate the basic ideas behind time-energy-uncertainty relations.

Definition 2. (Measurements, POVM)
In a very general sense a measurement can described by the following steps (see also Figure 1).

- We prepare a physical system. For better illustration, let's say that's an electron. The information about the configuration of that electron is contained in the state vector $|\psi\rangle$.
- We insert that electron into a measurement device.
- We get back a value. In the example of Figure 1, this value is either $1, 2$ or 3.
- If we repeat this measurement procedure with electrons that are prepared in the same way, which means their configuration is the same, we can get a different value for each measurement.
- We collect the measured values and their respective frequencies. Now we are able to write the empirical distribution function.

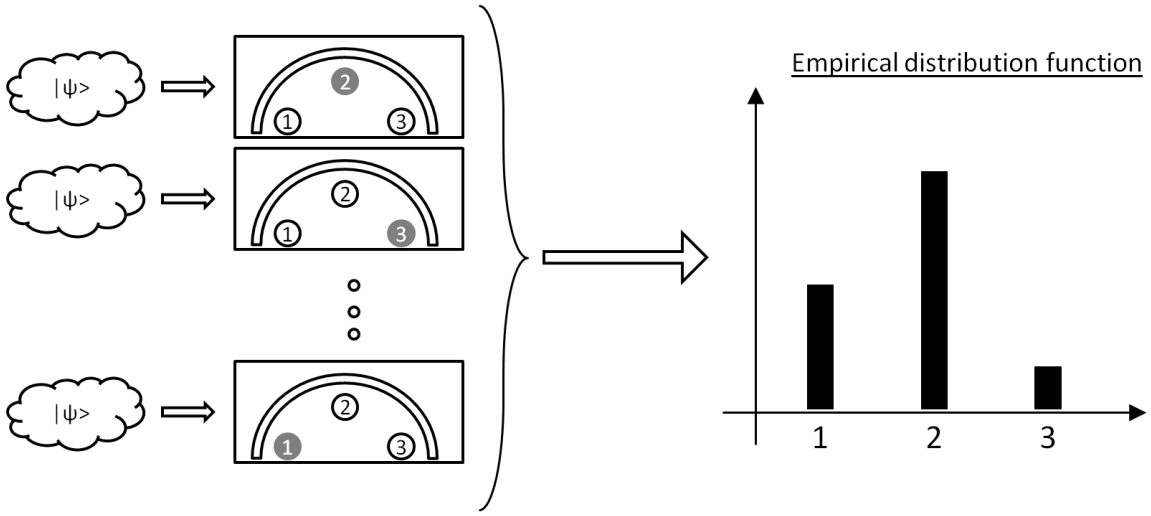

Figure 1: The prepared system $|\psi\rangle$ will be put into the measurement device. The device then will turn back a value. Iterated measurement of the 'same' prepared state will give an empirical distribution function.

This general case is modelled in Quantum mechanics by positive operator valued measure. So every possible value for a measurement result will get a corresponding positive hermitian matrix. A Positive Operator Valued Measure (POVM) on a measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a map $\mu : \mathcal{B}(\mathbb{R}) \to B(\mathcal{G})$ with values in the set of positive semi definite matrices on $\mathscr{H}$, such that

- For all $A \in \mathcal{B}(\mathbb{R}) : 0 \leq \mu(A) \leq \mathbb{1}_n$.
- $\mu(\mathbb{R}) = \mathbb{1}_n$
- For a sequence of pairwise disjoint sets $(A_n \in \mathcal{B}(\mathbb{R}))_{n\in\mathbb{N}}$

  $$\mu\left(\biguplus_{n=1}^{\infty} A_n\right) = \sum_{i=1}^{\infty} \mu(A_n)$$

  holds.

The idea is that for every effect there exists a matrix. In the example of Figure 1 a POVM would be the set of matrices $\{E_1, E_2, E_3\}$. From the matrices we can get the empirical distribution of that effect. The probability to measure effect 1 is then $\langle E_1\rangle := \langle\psi|E_1|\psi\rangle$. We write for $\langle\psi|A\psi\rangle$ also $\langle\psi|A|\psi\rangle$, so we can keep the graphical representation by bras and kets.

Definition 3. (Observable)
An observable is a special case of a POVM. Let $a_1, \ldots a_n$ be the n possible measured values for that property. Let further $A \in \mathbb{C}^{n\times n}$ be a hermitian matrix with eigenvalues $a_1, \ldots, a_n$.
A hermitian operator C suffices the following condition: $\langle\psi|C\phi\rangle = \langle C\psi|\phi\rangle$.
The eigenvectors (or eigenstates) of an observable form an orthonormal basis in the Hilbert space $\mathscr{H}$. Therefore by the spectral theorem,

$$A = \sum_i a_i P_i,$$

where a_i is an eigenvalue of A, and P_i is the orthogonal projection to the space spanned by the corresponding eigenvectors. We call this operator observable. To probability to measure a_i is then analogously to before $\langle\psi|P_i|\psi\rangle$. The expectation of the outcome of a measurement is then

$$\langle A\rangle := \sum_i a_i p_i = \sum_i a_i \langle\psi|P_i|\psi\rangle = \langle\psi|A|\psi\rangle,$$

where p_i is the probability to measure a_i.
After measuring a certain eigenvalue a_i the state is exactly in the normed state which lies in the *span* of corresponding eigenvector. So for observables, there is something like an update rule. That also means that for observables, there exist states for which a outcome a_k has probability 1. Namely exactly those vectors lying in the corresponding eigenspace. In general this doesn't apply to arbitrary POVM; there the state after the measurement is normally not known.

Definition 4. (Time evolution, Hamiltonian)
The evolution of that physical system in time is explained by the following equation

$$|\psi(t)\rangle = \exp(-itH)|\psi(0)\rangle, \tag{1}$$

where H is the so called Hamiltonian.
The Hamiltonian is a linear operator on $\mathscr{H}$ which corresponds to the total energy of a physical system. The Hamiltonian is hermitian.

Definition 5. (Commutator, Anti commutator, Standard deviation, covariance)
For two operators X, Y let

$$[X,Y] := XY - YX \qquad \{X,Y\}_+ := XY + YX,$$

be the commutator and anti commutator of X and Y.

We define the standard deviation of an operator A as

$$\Delta_\psi(A) := \left[\langle A\psi|A\psi\rangle - \langle\psi|A|\psi\rangle^2/\langle\psi|\psi\rangle\right]^{1/2} \overset{if\ \|\psi\|=1}{=} \left[\langle A^2\rangle - \langle A\rangle^2\right]^{1/2}.$$

For two observables A, B and for $\psi \in \mathscr{D}(AB) \cap \mathscr{D}(BA)$ we define their covariance as

$$\mathrm{cov}_\psi(A,B) := \langle\psi|\{A - a\mathbb{1}, B - b\mathbb{1}\}_+\psi\rangle/2,$$

where $a := \langle\psi, A\psi\rangle/\|\psi\|_2^2$ and analogously $b := \langle\psi, B\psi\rangle/\|\psi\|_2^2$.

2. Some auxiliary Lemmas

Theorem 1. *If A, B are Hermitian operators then for all $\psi \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$*

$$\|A\psi\|_2\|B\psi\|_2 \geq \frac{1}{2}\left[\langle\psi|\{A,B\}_+\psi\rangle^2 + |\langle\psi|[A,B]\psi|^2\rangle\right]^{1/2}, \tag{2}$$

where $\{A,B\}_+ = AB + BA$ and $[A,B] = AB - BA$. Equality holds in Eq.(2) iff $A\psi$ and $B\psi$ are proportional.

Corollary 1. (Robertson-Schrödinger)
If A, B are Hermitian operators then for all non-zero $\psi \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$

$$\Delta_\psi(A)\Delta_\psi(B) \geq \left[\mathrm{cov}_\psi(A,B)^2 + \frac{1}{4}|\langle\psi|\,[A,B]\,\psi\rangle|^2\right]^{1/2}, \tag{3}$$

where equality holds iff $(A - a\mathbb{1})\psi$ is a scalar multiple of $(B - b\mathbb{1})\psi$.
With $\mathrm{cov}_\psi(A,B) := \langle\psi|\{A - a\mathbb{1}, B - b\mathbb{1}\}_+\psi\rangle/2$, where $a := \langle\psi, A\psi\rangle/\|\psi\|_2^2$ and $b := \langle\psi, B\psi\rangle/\|\psi\|_2^2$.

3. Time-energy uncertainty relations

3.1 Mandelstam-Tamm inequality and its application

Theorem 2. ***(Mandelstam-Tamm inequality)***
Let A be an observable, $\mathscr{H}$ be a Hilbert space as before. If time evolution is governed by a Hamiltonian H (see (1)), then we have

$$\Delta_\psi(H)\Delta_\psi(A) \geq \frac{1}{2}\left|\frac{d\langle A\rangle}{dt}\right| \tag{4}$$

The Mandelstam-Tamm inequality yields that the variation of the distribution of an observable and the variation in the energy distribution can not both be small at the same time. The variation of an observable will be large, if the expected change in the observable is already large.

Definition 6. (Orthogonal states, survival probability)
We call two states ψ and ϕ orthogonal or perfectly distinguishable if $\langle\psi|\phi\rangle = 0$. For two orthogonal states exists a hermitian matrix E such that $\langle\psi|E|\psi\rangle = 1$ and $\langle\phi|E|\phi\rangle = 0$. So with the measurement of the effect described by E we can tell if the state was in ψ or not.
We define

$$A^0 := |\psi(0)\rangle\langle\psi(0)|.$$

This is the projection onto the space spanned by the initial state. We call

$$P(t) := \langle A^0\rangle_{\psi(t)}.$$

the survival probability. With this definition $P(0) = 1$. $\psi(t_0)$ is the state at time t_0 such that the state is perfectly distinguishable from the initial state. Then $P(t_0) = 0$. $P(t)$ gives the probability that the state at time t still lies within the space spanned by the initial state.
We see that $\Delta_\psi(A^0) = [P(t) - P(t)^2]^{1/2}$

Applying this notation to (4) gives the following result.

Corollary 2. (An application of the Mandelstam-Tamm inequality)

$$\Delta_\psi(H)t \geq \cos^{-1}\left(\sqrt{P(t)}\right) \tag{5}$$

With $t_{1/2}$ and t_0 as the shortest times according to (5) to drop to $1/2$ or 0 respectively yields

$$\Delta_\psi(H)t_{1/2} \geq \frac{\pi}{4} \qquad \Delta_\psi(H)t_0 \geq \frac{\pi}{2}$$

So if the physical system (think again of an electron) will quickly leave the initial state, then the variation in the energy will be large and vice versa.

Theorem 3. ***(Margolus-Levitin bound)***
Let t_0 be any time at which a positive semi-definite Hamiltonian H makes a state ψ with energy $\langle H \rangle$ evolve to an orthogonal state. Then

$$\langle H \rangle t_0 \geq \frac{\pi}{2}. \tag{6}$$

The Margolus-Levitin bound states that if a physical system changes quickly, then the expected total energy of that system is large.

3.2 Quantum Cramer-Rao bound

A more general question which arises of the previous framework is how accurate we can measure time. Assume we are given a state that evolved as $|\psi(t)\rangle = exp[-iHt]|\psi(0)\rangle$ for a fixed but unknown time t and we perform a measurement giving outcomes $\theta \in \mathbb{R}$ with probability density $p_t(\theta)$ and mean value $\langle \theta \rangle_t := \int \theta p_t(d\theta)$. The measurement is said to yield a locally unbiased estimator at t if $\langle \theta \rangle_t = t$, i.e. repeated measurements will asymptotically reveal the correct value of t. The rate at which this is approached is typically quantified in terms of the standard deviation $\Delta_t(\theta) := (\int p_t(d\theta)(\theta - t)^2)^{1/2}$.

Theorem 4. ***(Quantum Cramer-Rao bound)***
For any locally unbiased estimator θ for the time parameter t, the standard deviation $\Delta_t(\theta)$ of the measurement outcomes satisfies

$$\Delta_t(\theta)\Delta_\psi(H) \geq \frac{1}{2}. \tag{7}$$

The above Theorem shows that we can not measure time arbitrarily accurate.

References

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