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Differential Topology: Exercise Sheet 8

Exercises (for Feb. 6th and 7th)

8.1 Define $f, g : \mathbb{R}^3 :\to \mathbb{R}$ as

$$f(x, y, z) = x^3y^2z^2 + y^4x - 7xz^2 \tag{1}$$
$$g(x, y, z) = 2x^3z^8 - 5y^2z + yx^4 \ . \tag{2}$$

Prove, using differential topology, that f and g have a common non-zero zero, i.e that there is an $a \in \mathbb{R}^3 \setminus \{0\}$ such that $f(a) = g(a) = 0$.

Solution:
The maps $\tilde{f} := f|_{S^2}$ and $\tilde{g} := g|_{S^2}$ are odd smooth maps from S^2 to $\mathbb{R}$. Then, by (Corollary II to) Borsuk Ulam they have a common zero, i.e. there is $(x_0, y_0, z_0) \in S^2$ such that $f(x_0, y_0, z_0) = 0 = g(x_0, y_0, z_0)$.

8.2 Let $f : S^n \to S^n$ be a smooth map that carries the antipodal points to antipodal points. Compute $\deg_2(f)$.

Solution:
Antipodal points x, y satisfy $x = -y$ and are mapped to antipodal points $f(x), f(y)$ such that $f(x) = -f(y) = -f(-x)$. Hence f is smooth and odd, and by Borsuk Ulam (version II from the lecture) then $\deg_2(f) = 1$.

8.3 Construct a diffeomorphism $f : S^n \to S^n$ for which you prove that it is not smoothly homotopic to the identity map.

Solution:
Diffeomorphisms have degree ± 1 and the identity map has degree $+1$. The reflection

$$(x_1, x_2, \ldots, x_{n+1}) \mapsto (-x_1, x_2, \ldots, x_{n+1}) \tag{3}$$

was introduced in the lecture: it is a diffeomorphism of degree -1. Hence it is not smoothly homotopic to the identity map.

8.4 For any $n \in \mathbb{N}$ provide a smooth map $f : S^1 \to S^1$ with $\deg(f) = n$.

Solution:
Take the smooth map $\mathbb{C} \ni z \mapsto z^n \in \mathbb{C}$. As usual we regard S^1 as a subset of $\mathbb{R}^2$ and we identify the latter with $\mathbb{C}$ such that $x := \begin{pmatrix} \mathsf{Re}(z) \\ \mathsf{Im}(z) \end{pmatrix} \in S^1$ if and only if $z \in \mathbb{C}$ and $|z| = 1$.
Then the map $x \mapsto f(x) = \begin{pmatrix} \mathsf{Re}(z^n) \\ \mathsf{Im}(z^n) \end{pmatrix}$ maps from S^1 to S^1, is smooth and has degree n.

8.5 Let $f : S^n \to S^n$ be smooth and n even. Compute $\max_{x\in S^n} |\langle f(x), x\rangle|$ where x and $f(x)$ are regarded as unit vectors in $\mathbb{R}^{n+1}$.

Solution:
Since $x, f(x)$ are unit vectors we have the upper bound $\max_{x\in S^n} |\langle f(x), x\rangle| \le 1$. The smooth map f has either a fixed point or sends a point to its antipode (see lecture 14), i.e. there is a point $x_0 \in S^n$ such that $x_0 = \pm f(x_0)$. Therefore we find the lower bound $\max_{x\in S^n} |\langle f(x), x\rangle| \ge |\langle f(x_0), x_0\rangle| = 1$ which proves the claim.

8.6 Consider the following system of equations

$$2x + y + \sin(x+y) = 0 \tag{4}$$
$$x - 2y + \cos(x+y) = 0 \; . \tag{5}$$

Use the Euclidean degree to prove or disprove that there is a solution $(x_0, y_0) \in \mathbb{R}^2$ with $x_0^2 + y_0^2 < \frac{1}{4}$.

Solution:
Consider $H_t(z) := Az + tf(z)$ where $z = (x, y)$, $t \in [0, 1]$ and $A := \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$ and $f(z) := \begin{pmatrix} \sin(x+y) \\ \cos(x+y) \end{pmatrix}$. On the set $U := \{z \in \mathbb{R}^2 | \|z\| < \frac{1}{2}\}$ define $g : \bar{U} \to \mathbb{R}^2$ such that $g(z) = H_1(z)$. There is a solution to Eqs. (3) and (4) with norm smaller than $1/2$ if $0 \ne \deg(g, U, 0)$. We want to show that this equals $\deg(H_0, U, 0) = \mathsf{sgn}(\det(A)) = -1$. For that it suffices that $0 \notin H_t(\partial U)$ for all $t \in [0, 1]$. But for $\|z\| = \frac{1}{2}$ (i.e. for $z \in \partial U$) we have

$$\|H_t(z)\| \ge \|Az\| - t\|f(z)\| = \sqrt{5}\|z\| - t = \frac{\sqrt{5}}{2} - t > 0 \; . \tag{6}$$

Homotopy-invariance of the Euclidean degree then proves that there exists a solution.