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Differential Topology: Exercise Sheet 7

Exercises (for Jan. 30th and 31th)

7.1 Smooth homotopy

Two mappings $f, g : X \to Y$ are called smoothly homotopic if there exists a smooth map $F : X \times [0,1] \to Y$ with

$$F(x,0) = f(x) \qquad F(x,1) = g(x) \tag{1}$$

for all $x \in X$. Show that the relation of smooth homotopy is an equivalence relation.

Solution:

Reflexivity and symmetry are easy to see. To show that the relation is also transitive, we use a bump function. Let $\phi : [0,1] \to [0,1]$ be a smooth function that satisfies

$$\phi(t) = 0 \quad \text{for} \quad 0 \leq t \leq \tfrac{1}{3} \tag{2}$$

$$\phi(t) = 1 \quad \text{for} \quad \tfrac{2}{3} \leq t \leq 1 \tag{3}$$

(such functions exist, e.g. $\phi(t) = \frac{\xi(t-\frac{1}{3})}{\xi(t-\frac{1}{3})+\xi(\frac{2}{3}-t)}$ where $\xi(t) = 0$ for $t \leq 0$ and $\xi(t) = e^{-1/t}$ for $t > 0$). Given a smooth homotopy F between f and g, the formula $G(x,t) := F(x,\phi(t))$ defines another smooth homotopy G with

$$G(x,t) = f(x) \quad \text{for} \quad 0 \leq t \leq \tfrac{1}{3} \tag{4}$$

$$G(x,t) = g(x) \quad \text{for} \quad \tfrac{2}{3} \leq t \leq 1 \tag{5}$$

and for all $x \in X$. Now consider three smooth maps $f, g, h : X \to Y$ and assume that F_1 is a smooth homotopy between f and g and that F_2 is a smooth homotopy between g and h. Then $H : X \times [0,1] \to Y$ given by

$$H(x,t) = \begin{cases} G_1(x,2t) & \text{for } t \leq \frac{1}{2} \\ G_2(x,2t-1) & \text{for } t > \frac{1}{2} \end{cases} \tag{6}$$

is smooth by construction of G_1 and G_2 and satisfies $H(x,0) = G_1(x,0) = f(x)$ and $H(x,1) = G_2(x,1) = h(x)$. Hence H defines a smooth homotopy between f and h.

7.2 Stack of records theorem

Let $f : M \to N$ denote a smooth map between the smooth compact manifold $(M, \mathcal{A})$ and the smooth manifold $(N, \mathcal{B})$ with $\dim(N) = \dim(M)$. Suppose $y \in N$ is a regular value of f.

(a) Show that $f^{-1}(y) \subset M$ is a finite set.

(b) For $k \in N$ assume $f^{-1}(y) = \{x_1, \ldots, x_k\}$ for $x_i \in M$. Prove that there exist a neighborhood $U \subset N$ of y and disjoint open neighborhoods $V_i \subset M$ of x_i for all $i \in \{1, \ldots, k\}$ such that $f^{-1}(U) = V_1 \sqcup \cdots \sqcup V_k$ and such that $f|_{V_i} : V_i \to U$ is a diffeomorphism for all $i \in \{1, \ldots, k\}$.

Solution:

(a) Consider $x \in f^{-1}(y)$ and note that $\dim(M) = \dim(N)$ implies that the surjective map $df_x : T_x(M) \to T_y(N)$ is also injective as a linear map between vector spaces of the same dimension. Therefore by the inverse function theorem there exists an open neighborhood $V_x \subset M$ of $x \in M$ such that $f|_{V_x} : V_x \to f(V_x)$ is a diffeomorphism. This shows that $V_x \cap f^{-1}(y) = \{x\}$. As $f^{-1}(y)$ is closed (as a preimage of a continuous map of a closed set) and a subset of the compact manifold $(M, \mathcal{A})$ it is also compact. Therefore any cover $\bigcup_{x \in f^{-1}(y)} V_x$ with open sets V_x has a finite subcover, which shows that there are only finitely many points in $f^{-1}(y)$, because otherwise there exists a set V_x containing two distinct points $x, x' \in f^{-1}(y)$ which is a contradiction.

(b) Assume that there are $k \in \mathbb{N}$ points in $f^{-1}(y)$. Consider the open sets $\{V_{x_1}, \ldots, V_{x_k}\}$ defined in part (a) and shrink each of them until $V_{x_i} \cap V_{x_j} = \emptyset$ for all $i, j \in \{1, \ldots, k\}$. As the V_{x_i} are open sets $M\ (\bigcup_i V_{x_i})$ is a closed subset of a compact manifold M and therefore also compact. Using continuity of f this shows, that $f(M\ (\bigcup_i V_{x_i}))$ is also compact and in particular closed. By definition $y \notin f(M\ (\bigcup_i V_{x_i}))$ and thus there exists an open neighborhood $U \subset N$ of $y \in N$ such that $U \cap f(M\ (\bigcup_i V_{x_i})) = \emptyset$. This implies $f^{-1}(U) \subset \bigcup_i V_{x_i}$, which finishes the proof by taking sets $V_i = f^{-1}(U) \cap V_{x_i}$. With this definition we have $\sqcup_i V_i = f^{-1}(U) \cap \bigcup_i V_{x_i} = f^{-1}(U)$. Clearly $f|_{V_i}$ is a diffeomorphism as $V_i \subset V_{x_i}$ and also $V_i \cap V_j = \emptyset$ for all $i, j \in \{1, \ldots, k\}$.

7.3 Boundary theorem

Let M, N be smooth manifolds with $\dim M = \dim N$. Suppose that $M = \partial W$ for a compact manifold W and let $g : M \to N$ be a smooth map. Show that if g extends to a smooth map $G : W \to N$ then $\deg_2(g) = 0$.

Hint: You may use the fact that compact, one-dimensional manifolds have an even number of boundary points.

Solution:

Let $y \in N$ be a regular value, both for G and $g = \partial G$. Then, due to $\dim M = \dim N$, $G^{-1}(y)$ is a compact, one-dimensional manifold with boundary $\partial G^{-1}(y) = G^{-1}(y) \cap \partial W = g^{-1}(y)$. Using the hint, this set has an even number of elements, so $\deg_2(g) = 0$.

7.4 No-retraction theorem

Let W be a compact manifold with non-empty connected boundary $\partial W \neq \emptyset$. Show that there is no retraction of W to ∂W, i.e. there is no smooth map $f : W \to \partial W$ such that $f|_{\partial W} = \mathsf{id}_{\partial W}$.

Hint: Use the boundary theorem from the previous exercise.

Solution:

Assuming there is a smooth retraction with $f|_{\partial W} = \mathsf{id}_{\partial W}$, we conclude that $\deg_2(f|_{\partial W}) = 1$. But this contradicts the statement of the previous exercise.