Prof. Dr. M. Wolf
M. Heinze

Differential Topology: Exercise Sheet 6

Exercises (for Jan. 16th and 17th)

6.1 Application 2 of Brouwer's fixed point theorem: existence of a Nash equilibrium

In this exercise you will use Brouwer's fixed point theorem to show the existence of Nash equilibria which appear in game theory.

Formally, an n-player game is given by a tuple (n, f, S). To every player $i = 1, \dots, n$ we assign a non-empty, finite set of pure strategies S_i. Let $S = S_1 \times \cdots \times S_n$ be the set of all combinations of strategies. Player i may choose a (pure) strategy $x_i \in S_i$, this represents an option that the player may play. The payoff function $f : S \to \mathbb{R}^n$ is defined as $f(x) = (f_1(x), \dots, f_n(x))$ for $x = (x_1, \dots, x_n) \in S$. Note that the payoff $f_i(x)$ of player i may not only depend on his or her own choice x_i but also on the strategies of all other players.

We slightly extend this definition to *mixed strategies*: now, every player may choose a probability distribution over pure strategies. Each coordinate in a vector of a mixed strategy M_i is the probability of the corresponding pure strategy. E.g. the vector $(1, 0, \dots, 0)$ represents playing the first pure strategy in S_i. Then the payoff function $f : M \to \mathbb{R}^n$ is a convex combination of the payoff of pure strategies.

For a strategy $x \in M$ let us introduce the notation $x_{-i} \in M_1 \times \cdots M_{i-1} \times M_{i+1} \times \cdots \times M_n$ to denote the strategies of all players but player i.

A strategy $x^* \in M$ is a *Nash equilibrium* of a game (n, f, M) if no deviation of this strategy by a single player can increase the payoff function of this player, that is for all $i = 1, \dots, n$ and any $x_i \in M_i$ we find

$$f_i(x^*) \geq f_i(x^*_{-i}, x_i) \ . \tag{1}$$

(a) Define the gain function for player i quantifying how much higher the payoff is of the j-th pure strategy in S_i over a mixed strategy $x_i \in M$ (given the mixed strategies of the other players), that is $g_{i,j} : M \to M$ and

$$g_{i,j}(x) = \max\{0, f_i(e_j, x_{-i}) - f_i(x)\} \ . \tag{2}$$

For $i = 1, \dots, n$ define $b_i : M \to M_i$ as

$$b_i(x) = \left(\frac{x_{i,1} + g_{i,1}(x)}{1 + \sum_{j=1}^{|S_i|} g_{i,j}(x)}, \dots, \frac{x_{i,|S_i|} + g_{i,|S_i|}(x)}{1 + \sum_{j=1}^{|S_i|} g_{i,j}(x)} \right) \ . \tag{3}$$

Argue that for $x \in M$ the function $b(x) = (b_1(x), \dots, b_n(x))$ maps to valid mixed strategies and that it is continuous.

(b) We define the best response of a given strategy as the best option that player i may play given the strategies of the other players, i.e. for $x \in M$ the best response for player i is $r_i(x_{-i}) = \{y \in M_i | f_i(x_{-i}, y) = \max_{z \in M_i} f_i(x_{-i}, z)\}$.
Show that a pure strategy attains maximal payoff, that is for every strategy $x \in M$ and all $i = 1, \ldots, n$ there exists a pure strategy $y_i \in S_i$ in $r_i(x_{-i})$.

(c) Apply Brouwer's fixed point theorem to $b : M \to M$ from (a). Combine this with (b) to show the existence of a Nash equilibrium.

Solution:

(a) To show that $b_i : M \to M_i$ is well-defined, note that $x_{i,j} \in [0,1]$. First observe that therefore for all j the coordinates of $b_i(x)$ satisfy

$$\frac{x_{i,j} + g_{i,j}(x)}{1 + \sum_{j=1}^{|S_i|} g_{i,j}(x)} \geq 0 \ . \tag{4}$$

Additionally, we find that they sum up to 1, i.e.

$$\sum_{j=1}^{|S_i|} \frac{x_{i,j} + g_{i,j}(x)}{1 + \sum_{k=1}^{|S_i|} g_{i,k}(x)} = \frac{\sum_{j=1}^{|S_i|} x_{i,j} + \sum_{j=1}^{|S_i|} g_{i,j}(x)}{1 + \sum_{k=1}^{|S_i|} g_{i,k}(x)} = \frac{1 + \sum_{j=1}^{|S_i|} g_{i,j}(x)}{1 + \sum_{k=1}^{|S_i|} g_{i,k}(x)} = 1 \ . \tag{5}$$

Combing both equations, we find that

$$0 \leq \frac{x_{i,j} + g_{i,j}(x)}{1 + \sum_{j=1}^{|S_i|} g_{i,j}(x)} \leq 1 \tag{6}$$

We conclude that $b_i(x) \in M_i$ and that therefore $b : M \to M$ is well defined.
Let us now show that $b : M \to M$ is a continuous map. We know that for a mixed strategy $x \in M$ the payoff $f_i(x)$ is defined as a linear combination of the pure strategies' payoffs $f_i(e_j)$ (for a finite number $j = 1, \ldots, |S_i|$). Considering changes in the probabilities with which the pure strategies e_j are played, the payoff function f_i is continuous. Then also the gain functions $g_{i,j}$ are continuous as the difference of continuous functions. Continuity of b follows because each of its coordinates is a linear combination of $g_{i,j}$.

(b) Let us consider the set of mixed strategies $x = (x_1, \ldots, x_n) \in M$ played by n players and vary the strategy of player i. Since there are only finitely many pure strategies in $|S_i|$, it follows that there must be one best pure strategy e^* for this player, i.e.

$$f_i(x_{-i}, e^*) = \max_{j=1,\ldots,|S_i|} f_i(x_{-i}, e_j) \tag{7}$$

where the e_j are unit vectors. Now consider any mixed strategy $y_i \in M_i$ of this player. Then there are $\lambda_j \in [0,1]$ (for $j = 1, \ldots, |S_i|$) with $\sum_{j=1}^{|S_i|} \lambda_j = 1$ that correspond to the probabilities with which the pure strategies e_j are played in y_i. The payoff function of player i playing strategy y_i (assuming again that the other players stay with their strategies x_{-i}) is

$$f_i(x_{-i}, y_i) = \sum_{j=1}^{|S_i|} \lambda_j f_i(x_{-i}, e_j) \leq \sum_{j=1}^{|S_i|} \lambda_j f_i(x_{-i}, e^*) = f_i(x_{-i}, e^*) \tag{8}$$

where we used equation (7). By definition of the best response $r_i(x_{-i})$ we conclude that $e^* \in r_i(x_{-i})$.

(c) By (a) we know that $b : M \to M$ is a well-defined continuous map. The finiteness of the sets of pure strategies $S_1, \dots, S_n$ and the definition of M via probabilities of pure strategies imply compactness. Hence by Brouwer's fixed point theorem there is a fixed point $x^* \in M$ such that $b(x^*) = x^*$. This implies for all i that

$$x_i^* = \left(\frac{x_{i,1}^* + g_{i,1}(x)}{1 + \sum_{j=1}^{|S_i|} g_{i,j}(x^*)}, \dots, \frac{x_{i,|S_i|}^* + g_{i,|S_i|}(x^*)}{1 + \sum_{j=1}^{|S_i|} g_{i,j}(x^*)} \right) \tag{9}$$

Then for all $i = 1, \dots, n$ and all $j = 1, \dots, |S_i|$, $g_{i,j}(x^*) = 0$, which means that for every player i there is no pure strategy e_j with a higher payoff than the x^*, i.e. $f_i(x^*) \geq f_i(x_{-i}^*, e_j)$. Now (b) implies that the maximal payoff is attained by at least one pure strategy. Combining these two statements, we conclude that no mixed strategy can have higher payoff than x^* which is exactly the definition of the Nash equilibrium.

6.2 Embedding of projective spaces

Show that the function $f : \mathbb{R}P^2 \to \mathbb{R}^4$ defined by

$$f\big(q(x_1, x_2, x_3)\big) = (x_1^2 - x_2^2, x_1x_2, x_1x_3, x_2x_3)$$

is an embedding of $\mathbb{R}P^2$ into $\mathbb{R}^4$. Here, q is the canonical projection onto equivalence classes. We use the definition of $\mathbb{R}P^2$ as $S^2/\sim$ where we identify antipodal points (see exercise 2.1). You may use without proof that q is locally a diffeomorphism (This could be verified using charts).

Solution:

We have to show that f is an immersion which maps $\mathbb{R}P^2$ homeomorphically to $f(\mathbb{R}P^2) \subset \mathbb{R}^4$. It is clear that f is well-defined since every coordinate is a quadratic function, so the signs of the entries cancel.

We start by showing that f is an immersion. As q is locally a diffeomorphism, d_pq is surjective and we can consider the map $g = f \circ q : S^2 \to \mathbb{R}^4$, where we identify S^2 as a submanifold of $\mathbb{R}^3$. Furthermore, f is smooth since g is. We compute

$$d_xg = \begin{pmatrix} 2x_1 & -2x_2 & 0 \\ x_2 & x_1 & 0 \\ x_3 & 0 & x_1 \\ 0 & x_3 & x_2 \end{pmatrix}$$

Note that we cannot just check whether d_xg has full rank, since the tangent space at a point of S^2 is only two-dimensional. Clearly, any two non-zero elements of the vectors $t_1 = (x_2, -x_1, 0)$, $t_2 = (x_3, 0, -x_1)$ and $t_3 = (0, x_3, -x_2)$ span T_xS^2. We compute

$$\begin{aligned} d_xg(t_1) &= (4x_1x_2, x_2^2 - x_1^2, x_2x_3, -x_1x_3) \\ d_xg(t_2) &= (2x_1x_3, x_2x_3, x_3^2 - x_1^2, -x_1x_2) \\ d_xg(t_3) &= (-2x_2x_3, x_1x_3, -x_1x_2, x_3^2 - x_2^2) \end{aligned}$$

A moment of thought shows that whenever t_i and t_j are non-vanishing, $i \neq j$, then $d_xg(t_i)$ and $d_xg(t_j)$ are linearly independent. Therefore, the map d_xg is injective and f is an immersion.

This leaves to prove that $f : \mathbb{R}P^2 \to f(\mathbb{R}P^2)$ is a homeomorphism. Since $\mathbb{R}P^2$ is compact, it suffices to show that f is an injective immersion. For $f(q(x)) = f(q(x'))$, it follows that $x_i x_j = x'_i x'_j$. If $x_i \neq 0$, this implies that $q(x) = q(x')$. If for example $x_3 = 0$ and x_1, $x_2 \neq 0$, then $(x'_1, x'_2) = \pm(x_1, x_2)$ or $\pm(x_2, x_1)$ and the sign of $x_1^2 - x_2^2$ determines which is the case. The same works for the other two cases. If two variables are 0, then the last must be ± 1. Thus, f is injective and therefore an embedding. Remark: It is also true that $\mathbb{R}P^2$ cannot be embedded into $\mathbb{R}^3$.