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Sheet 4

Differential Topology: Exercise Sheet 4

Exercises (for Dec. 5th and 6th)

4.1 Different definitions of tangent space

Consider the geometrically more intuitive definition of a tangent space from the motivating example in the lecture: Let M be a smooth submanifold embedded in some $\mathbb{R}^n$ and define

$$\vec{T}_x M := \{v \in \mathbb{R} \mid \gamma \in C^\infty\big((-1,1), M\big), \gamma(0) = x, \gamma'(0) = v\} \tag{1}$$

Show that the the vector spaces $T_x M$ and $\vec{T}_x M$ are isomorphic, i.e, that the map $T_x M \to \vec{T}_x M$, $[\gamma] \mapsto \gamma'(0)$ is a vector space isomorphism.

Solution:

To show that the map $[\gamma] \mapsto \gamma'(0)$ is a vector space isomorphism we use the submanifold property, i.e., for every chart $(U, \phi) \in \mathcal{A}$ of M such that $x \in U$ there is a set $V \in \mathbb{R}^n$ and a local diffeomorphism $\Phi : V \to \mathbb{R}^n$ such that

$$\Phi^{-1}(\{x \in \mathbb{R}^n \mid x_{m+1} = \cdots = x_n = 0\}) = V \cap M = U \tag{2}$$

Here we have $\Phi(y) = i \circ \phi(y)$ for all $y \in U$ where $i : \mathbb{R}^m \to \mathbb{R}^n$ is the inclusion map $i(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0)$.

By definition of the equivalence class $[\gamma]$ we can write $[\gamma] = [\phi^{-1}(\phi(x) + t\xi)]$ for some $\xi = (\phi \circ \gamma)'(0) \in \mathbb{R}^m$ (as shown in the lecture, in fact, the map $\mathbb{R}^m \ni \xi \mapsto [\phi^{-1}(\phi(x) + t\xi)] \in T_x M$ is a vector space isomorphism between $\mathbb{R}^m$ and $T_x M$). Setting $\lambda \in \mathbb{R}$ and $\xi_1 = (\phi \circ \gamma_1)'(0)$, $\xi_2 = (\phi \circ \gamma_2)'(0)$ and using $\phi^{-1} = \Phi^{-1} \circ i$, this implies

$$\begin{aligned}
\lambda[\phi^{-1}(\phi(x) + t\xi_1)] + [\phi^{-1}(\phi(x) + t\xi_2)] &= [\Phi^{-1} \circ i(\phi(x) + t(\lambda\xi_1 + \xi_2)] \\
&=: [\gamma] \mapsto \gamma'(0) \\
&= \left(\Phi^{-1} \circ i \circ \phi \circ \gamma\right)'(0) \\
&= (d\Phi^{-1})(\phi \circ \gamma(0))\ i(\phi \circ \gamma)'(0) \\
&= (d\Phi^{-1})(\phi(x))\ i(\lambda\xi_1 + \xi_2)
\end{aligned}$$

where we have $\phi \circ \gamma(0) = \phi(x)$. Hence the two vector spaces $T_x M$ and $\vec{T}_x M$ are isomorphic.

4.2 Lie group actions

Let G denote a group and X an arbitrary set. A (left) group action of G on X is a map

$$\alpha : G \times X \to X$$

which has the properties

(i) $\alpha(e,\cdot) = \mathrm{id}_X$ for the unit element $e \in G$

(ii) $\alpha(g, \alpha(h,x)) = \alpha(gh, x)$ for all $g, h \in G$, $x \in X$.

In the case where G is a Lie group and X is a smooth manifold, we call a Lie group action a smooth group action of G on X, i.e. α is a smooth map such that $\alpha(g,\cdot) : X \to X$ is a diffeomorphism for every $g \in G$. In the following let α denote a Lie group action of the Lie group G on the smooth manifold X.

(a) Show that $\alpha(\cdot, x) : G \to X$ has constant rank for all $x \in X$. (Hint: Use that the left-multiplication $L_g : G \to G$ $L_g(h) = gh$ is a diffeomorphism.)

(b) Let $G_x = \{h \in G | \alpha(h,x) = x\}$ be called the stabilizer of $x \in X$ and let $U \cdot G_x = \{g \cdot h | g \in U, h \in G_x\}$. Show that $\alpha(U \cdot G_x, x) = \alpha(U, x)$ and that for $U \subset G$ open, $U \cdot G_x$ is open.

(c) Let G be a compact Lie group, $U \subset G$ open and $x \in X$. Show that $\alpha(U, x)$ is open.

(d) Let G be a compact Lie group. Show that the orbit of $x \in X$ under the action α of G on X, defined as

$$\mathcal{O}_x = \alpha(G, x),$$

is a smooth submanifold of X.

(e) Consider the special case $G = U(n)$ of $n \times n$ unitary matrices and $X = \mathcal{H}_n$ of $n \times n$ Hermitian matrices. Show that the map $\alpha : U(n) \times \mathcal{H}_n \to \mathcal{H}_n$ defined by $\alpha(U, A) = UAU^*$ is a Lie group action.

(f) Show that the unitary equivalence orbit of $A \in \mathcal{H}_n$, denoted by $\mathcal{O}_A = \{UAU^* | U \in U(n)\}$ is a smooth manifold.

Solution:

(a) Let p, $q \in G$ and let $g \in G$ such that $q = g \cdot p$. Let $L_g : G \to G$ be the left-multiplication by g. Since $L_g = \mu \circ \iota_g$, where $\iota_g : G \to G \times G$, $\iota_g(h) = (g, h)$ is the inclusion map and $\mu : G \times G \to G$ is the multiplication map, L_g is smooth. As it has a smooth inverse $L_{g^{-1}}$, it is a diffeomorphism. Consider the equation

$$\alpha(g,\cdot) \circ \alpha(\cdot, x) = \alpha(\cdot, x) \circ L_g$$

which is true by the properties of the group action. Differentiating at p yields

$$d_{\alpha(p,x)}\alpha(g,\cdot) \cdot d_p\alpha(\cdot, x) = d_q\alpha(\cdot, x) \cdot d_pL_g.$$

As $\alpha(g,\cdot) : X \to X$ and L_g are diffeomorphisms, it follows that $\mathsf{rank}(d_p\alpha(\cdot, x)) = \mathsf{rank}(d_q\alpha(\cdot, x))$ for all p, $q \in G$. Therefore, $\alpha(\cdot, x) : G \to X$ has constant rank.

(b) The first assertion holds that for $g \in U$, $h \in G_x$, we have $\alpha(g, x) = \alpha(g, \alpha(h, x)) = \alpha(gh, x)$. Analogously to left-multiplication, right-multiplication $R_g : G \to G$ is also a diffeomorphism. Thus $U \cdot h = R_h(U)$ is open for all $h \in G_x$ and the second assertion follows by taking the union of all such sets.

(c) Note that since $G \setminus (U \cdot G_x)$ is closed and G is compact, the former set is compact and $\alpha(G \setminus (U \cdot G_x), x)$ is compact as well, since $\alpha(\cdot, x)$ is continuous. It is also closed, since X is Hausdorff. If we could show that

$$\alpha(G, x) \setminus \alpha(U, x) = \alpha(G \setminus (U \cdot G_x), x), \tag{3}$$

we would be done. We show first $\alpha(G \setminus (U \cdot G_x), x) \cap \alpha(U, x) = \emptyset$. This can be seen as follows. Let $g \in U$, $h \in G \setminus U \cdot G_x$ such that $\alpha(g, x) = \alpha(h, x)$. This implies $g^{-1}h \in G_x$ and hence $h \in U \cdot G_x$, which is a contradiction. As $\alpha(G, x) = \alpha(G \setminus (U \cdot G_x), x) \cup \alpha(U, x)$, Equation (3) holds. As G is Hausdorff and $\alpha(G, x) \setminus \alpha(U, x)$ is compact, $\alpha(U, x)$ is open.

(d) In part (a) of this exercise, we have shown that $\alpha(\cdot, x)$ has constant rank. Let $\mathcal{A}$ be the atlas of G and $\mathcal{B}$ the atlas on X. Consider an arbitrary $y \in \mathcal{O}_x$, i.e. there is a $g \in G$ such that $y = \alpha(g, x)$. By the constant rank theorem, there are $(U, \varphi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $g \in U$ and $\alpha(U, x) \subset V$ and $\psi \circ \alpha(\cdot, x) \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^{\dim X}$ is the canonical embedding

$$\psi \circ \alpha(\cdot, x) \circ \varphi^{-1}(x_1, \ldots, x_{\dim G}) = (x_1, \ldots, x_{\mathsf{rank}\alpha(\cdot, x)}, 0, \ldots, 0)$$

As $\alpha(U, x)$ is open by part (c) of this exercise, $(\alpha(U, x), \psi)$ is an adapted chart and the construction makes $\mathcal{O}_x$ into a submanifold of X.

(e) The Hermitian matrices are smooth manifold of dimension n^2, since they form a real vectorspace of this dimension and choosing a basis gives a global chart into $\mathbb{R}^{n^2}$. That $U(n)$ is a Lie group follows by the same reasoning as for $O(n)$ (see example in Lecture 5). The properties (i) and (ii) are clearly satisfied. We can write $\alpha(U, A) = \mu(\mu(U, A), \iota(U))$ because $U^* = U^{-1}$, which is a composition of smooth maps. Here, μ is the matrix multiplication in $\mathbb{C}^{n \times n}$, which is clearly smooth, and ι the inversion on $U(n)$. As $\alpha(U, \cdot)$ has a smooth inverse map $\alpha(U^*, \cdot)$, it is a diffeomorphism and α is a Lie group action.

(f) We only have to show that $U(n)$ is compact. By the Heine-Borel theorem it is enough to show that $U(n)$ is closed and bounded. Boundedness follows since for the operator norm we have $||U||_\infty = 1$. For closedness, note that $U^*U = \mathsf{id}$ defines $U(n)$, thus $U(n)$ is the solution set to a system of polynomial equations and therefore closed.