Differential Topology: Exercise Sheet 2

Exercises (for Nov. 7th and 8th)

2.1 Real projective space

Consider the equivalence relation

$$x \sim y \qquad :\Leftrightarrow \qquad \exists \lambda \in \mathbb{R}\backslash\{0\} : x = \lambda y \tag{1}$$

on $\mathbb{R}^{n+1}\backslash\{0\}$. Then $\mathbb{R}P^n := (\mathbb{R}^{n+1}\backslash\{0\})/\sim$ is called the **(real) projective space.**

(a) Show that $\mathbb{R}P^n$ is homeomorphic to $S^n/\sim$ where the relative topology and the equivalence relation

$$x \sim y :\Leftrightarrow x = \pm y \tag{2}$$

on the unit sphere $S^n \subset \mathbb{R}^{n+1}$ are used. This provides an alternative definition of $\mathbb{R}P^n$.

In the following, we show that $\mathbb{R}P^n$ is a smooth n-manifold:

(b) Show that $\mathbb{R}P^n$ is Hausdorff.

(c) Show that $\mathbb{R}P^n$ is second countable.

(d) Define $U_j := \{x \,| x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}, x_j \neq 0\}$ and let $q : \mathbb{R}\backslash\{0\} \to \mathbb{R}P^n$ be the quotient map associated to the equivalence relation (1). Show that $\{q(U_j)\}_{j=1}^n$ is an open cover of $\mathbb{R}P^n$, and that each $q(U_j)$ is homeomorphic to a subset of $\mathbb{R}^n$.

(e) Show that $\mathbb{R}P^n$ is a smooth manifold. Show that the quotient map $q : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}P^n$ used to define $\mathbb{R}P^n$ is smooth.

Solution:

(a) In this exercise, we will construct a homemorphism between $\mathbb{R}P^n$ and $S^n/\sim$. Let us first show the following:

Lemma 1 *Let $f : X \to Y$ for some topological spaces X, Y, $\sim$ denote an equivalence relation on X and q be the quotient map of the quotient space $X/\sim$. Let f be constant on each equivalence class, such that it induces a map $\overline{f} : X/\sim \to Y$ by $\overline{f}(q(p)) = f(p)$ for all $p \in X$. The induced map $\overline{f} : X/\sim \mapsto Y$ is continuous if and only if the map $f : X \mapsto Y$ is continuous.* □

Proof : Let $\overline{f}$ be continuous. Since q is continuous by definition, also $f = \overline{f} \circ q$ is continuous as a composition of continuous functions. For the converse direction, assume that f is continuous. Let $U \subset Y$ be an arbitrary open set. By assumption, $f^{-1}(U) = q^{-1}(\overline{f}^{-1}(U))$ is open and by the definition of the quotient topology, also $\overline{f}^{-1}(U)$ is open. Since U was arbitrary, it follows that $\overline{f}$ is continuous. ■

We consider the following situation:

$$\begin{array}{ccc} \mathbb{R}^{n+1}\setminus\{0\} & \xrightarrow{f} & S^n \\ \downarrow q & & \downarrow q' \\ \mathbb{R}P^n & \xrightarrow{\overline{f_{q'}}} & S^n/\sim \end{array}$$

Let $f : \mathbb{R}^{n+1} \setminus \{0\} \mapsto S^n$ be defined by

$$f(x) = \frac{x}{||x||}$$

This is clearly a continuous function. Let $q' : S^n \mapsto S^n/\sim$ be the projection onto equivalence classes, which is also continuous. Then $q' \circ f$ is continuous as well. We show that $q' \circ f$ is constant on equivalence classes. Let q the projection map $q : \mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{R}P^n$. Let y, $z \in \mathbb{R}^{n+1}\setminus\{0\}$ such that $q(y) = q(z)$. Then $\exists\lambda \in \mathbb{R}\setminus\{0\}$ such that $\lambda y = z$. Since

$$f(z) = \frac{\lambda y}{|\lambda|||y||} = \text{sgn}(\lambda)f(y),$$

it follows that $f(y) \sim f(z)$ and hence $q'\circ f(y) = q'\circ f(z)$. By the above Lemma, $q'\circ f$ induces an $\overline{f_{q'}} : \mathbb{R}P^n \mapsto S^n/\sim$, which is continuous since $q'\circ f$ is. To find an inverse, consider $g : S^n \mapsto \mathbb{R}^{n+1}\setminus\{0\}$ given by $g(x) = x$. This is clearly continuous, as is $q\circ g$. We show again that $q \circ g$ is constant on equivalence classes. Let u, $v \in S^n$ such that $q'(u) = q'(v)$. Then $u = \pm v$ and hence also $g(u) \sim g(v)$, such that $q\circ g(u) = q\circ g(v)$. Hence $q \circ g$ induces a continuous $\overline{g_q} : S^n/\sim \to \mathbb{R}P^n$. We are left with showing that indeed $\overline{g_q}$ is an inverse of $\overline{f_{q'}}$. However, since

$$\overline{f_{q'}} \circ \overline{g_q}(q'(x)) = q' \circ f \circ g(x) = q'(x/||x||) = q'(x),$$

because $||x|| = 1$ for $x \in S^n$ and

$$\overline{g_q} \circ \overline{f_{q'}}(q(y)) = q \circ g \circ f(y) = q(y/||y||) = q(y),$$

we have verified $\overline{f_{q'}} \circ \overline{g} = \text{id}_{S/\sim}$ and $\overline{g} \circ \overline{f_{q'}} = \text{id}_{\mathbb{R}P^n}$. Thus $\overline{f_{q'}}$ is the desired homeomorphism.

(b) Take two points x, $y \in \mathbb{R}P^n$ such that $x \neq y$. Since we have shown in part 3 of this exercise that $\mathbb{R}P^n$ and $S^n/\sim$ are homeomorphic, consider corresponding s_x, $s_y \in S^n$ such that $\overline{g_q} \circ q'(s_x) = x$ and the same for s_y and y ($\overline{g_q}$ given as in the last exercise). Since S^n is Hausdorff (since it can be made into a metric space), we can find neighborhoods U_x, U_y of s_x and s_y, respectively, such that

$$U_x \cap U_y = \emptyset,\ U_x \cap -U_y = \emptyset, -U_x \cap U_y = \emptyset \text{ and } -U_x \cap -U_y = \emptyset,$$

where $-U_x = \{s \in S^n| - s \in U_x\}$ and likewise for $-U_y$. This is true since we can construct neighborhoods for each relation (using that $s_x \notin \{\pm s_y\}$ as $x \neq y$) and consider intersections to find neighborhoods which satisfy all the equations at once. Then also $\overline{g_q}(q'(U_x))$ and $\overline{g_q}(q'(U_y))$ are disjoint in $\mathbb{R}P^n$, since for $p \in \overline{g_q}(q'(U_x)) \cap \overline{g_q}(q'(U_y))$, we would have $\overline{f_{q'}}(p) \in q'(U_x) \cap q'(U_y)$, which means that $\overline{f_{q'}}(p)$ would be contained in one of the intersections above, which were constructed to be empty. Furthermore, $q'(U_x)$ and $q'(U_y)$ contain open sets, since $-V \subset S^n$ is open if and only if V is and $q'(V)$ is therefore open for every open set $V \subset S^n$. Thus, $\overline{g_q}(q'(U_x))$ and $\overline{g_q}(q'(U_y))$ are disjoint neighborhoods of x and y, respectively.

(c) We will exploit that $\mathbb{R}^{n+1}$ is second countable. Let $U \subset \mathbb{R}^{n+1} \setminus \{0\}$ be open and define $\lambda U = \{\lambda u \in \mathbb{R}^{n+1} \setminus \{0\} | u \in U\}$. Obviously, λU is open for all $\lambda \in \mathbb{R} \setminus \{0\}$. Therefore, $q^{-1}(q(U)) = \bigcup_{\lambda \in \mathbb{R} \setminus \{0\}} \lambda U$ is open, which implies that $q(U)$ is open as well. Let $\{U_i\}_{i \in \mathbb{N}}$ be a countable basis of $\mathbb{R}^{n+1}$. Let $V \subset \mathbb{R}P^n$ be open. Then $q^{-1}(V)$ is open and there is an index set $\mathcal{I} \subset \mathbb{N}$ such that $q^{-1}(V) = \bigcup_{i \in \mathcal{I}} U_i$. Then also $V = \bigcup_{i \in \mathcal{I}} q(U_i)$ and we have already argued that $q(U_i)$ is open for all $i \in \mathbb{N}$. Therefore, $\{q(U_i)\}_{i \in \mathbb{N}}$ is a countable basis for the topology on $\mathbb{R}P^n$.

(d) We have already seen in part (b) of this exercise that $q(U_j)$ is open because U_j is. Furthermore, $\bigcup_{i=1}^n q(U_i) = \mathbb{R}P^n$, since $\bigcup_{i=1}^n U_i = \mathbb{R}^{n+1} \setminus \{0\}$. Define $\Phi_i : U_i \mapsto \mathbb{R}^n$ by

$$\Phi_i(x) = (x_1/x_i, \ldots, \hat{x}_i, \ldots, x_{n+1}/x_i).$$

This is constant on equivalence classes and therefore induces a map $\overline{\Phi}_i : q(U_i) \mapsto \mathbb{R}^n$ by part 2. $\overline{\Phi_i}$ is continuous because Φ_i is. We can define a map $\Psi_i^{-1} : \Phi_i(U_i) \mapsto U_i$ given by

$$\Psi_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_n).$$

Then $q \circ \Psi_i$ is an inverse for $\overline{\Phi}_i$ and as a composition of continuous maps it is again continuous. Thus $q(U_i)$ and $\overline{\Phi}_i(q(U_i)) \subset \mathbb{R}^n$ are homeomorphic.

(e) We already know that $\mathbb{R}P^n$ is a topological manifold, thus we need to find a smooth structure. Let q be the projection on equivalence classes and define again $q(U_i) = \{q(x_1, \ldots, x_{n+1}) | x_i \neq 0\}$ and $\overline{\Phi}_i : q(U_i) \to \mathbb{R}^n$ as

$$\overline{\Phi}_i(q(x_1, \ldots, x_{n+1})) = \left[\frac{x_1}{x_i}, \ldots, \hat{x}_i, \ldots \frac{x_{n+1}}{x_i}\right]$$

($\hat{x}_i$ means that x_i is omitted) with inverse function

$$\overline{\Phi}_i^{-1} : (y_1, \ldots, y_n) \mapsto q(y_1, \ldots, y_{i-1}, 1, y_i, \ldots, y_n)$$

We have already seen that these definitions give us charts, so we have to see that the transition functions are smooth. For simplicity, let us consider $q(U_1) \cap q(U_2)$. Then for $x \in \overline{\Phi}_1(q(U_1) \cap q(U_2))$

$$\overline{\Phi}_2 \circ \overline{\Phi}_1^{-1}(x) = \left[\frac{1}{x_1}, \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\right].$$

This is smooth since $x_1 \neq 0$. On any other $q(U_i) \cap q(U_j)$, analogous statements can be made. Hence this defines a smooth structure.

2.2 Boundary and interior of a topological manifold

Let M be an n-dimensional topological manifold with non-trivial boundary ∂M. Show that

(a) $\mathsf{int}(M)$ is an n-dimensional manifold,

(b) ∂M is an $(n-1)$-dimensional manifold.

Solution:

(a) For $x \in \mathsf{int}(M)$ exists an open neighborhood $U \subset M$ of x that is homeomorphic to an open subset $V \subset \mathbb{R}^n_+$ via a homeomorphism $\phi : U \to V$. By definition of an interior point, we have that $\phi(x) = y \in \mathbb{R}^n_+$ where $y_n > 0$, i.e. there exists an open neighborhood $V' \subset \mathbb{R}^n_+ \setminus \{y | y_n = 0\}$ of y such that $(\phi^{-1}(V'), \phi|_{V'})$ gives a chart at x. As this works for arbitrary $x \in \mathsf{int}(M)$ the proof is done.

(b) Let $x \in \partial M$. By definition of a manifold with boundary, there is an open neighborhood $U \subset M$ of x that is homeomorphic to an open subset $V \subset \mathbb{R}^n_+$ via a homeomorphism $\phi : U \to V$. Since $x \in U \cap \partial M$ we have $\phi(x) \in V \cap \partial \mathbb{R}^n_+$. The restriction of the homeomorphism ϕ to $\phi^{-1}(V \cap \partial \mathbb{R}^n_+) = U \cap \partial M$ is again a homeomorphism $\phi|_{\phi^{-1}(V \cap \partial \mathbb{R}^n_+)} : \phi^{-1}(V \cap \partial \mathbb{R}^n_+) \to V \cap \partial \mathbb{R}^n_+$. Since $V \cap \partial \mathbb{R}^n_+$ is open in the subspace topology on $\partial \mathbb{R}^n_+$, it is an open neighborhood of $\phi(x) \in$. Hence we showed that the open neighborhood $U \cap \partial M$ $(=\phi^{-1}(V \cap \partial \mathbb{R}^n_+))$ of $x \in \partial M$ is homeomorphic to an open subset of $\partial \mathbb{R}^n_+$. The latter is defined as $\{x \in \mathbb{R}^n \mid x_n = 0\}$, i.e. isomorphic to $\mathbb{R}^{n-1}$.

2.3 Examples of differentiable manifolds

(a) Suppose M_j are smooth m_j-manifolds, for $j = 1, 2$. Show that there is a smooth structure on $M_1 \times M_2$ such that $M_1 \times M_2$ is a smooth $(m_1 + m_2)$-manifold, and the canonical projections $\pi_j : M_1 \times M_2 \to M_j$ are smooth, for $j = 1, 2$.

(b) Show that any open subset of a smooth manifold is again a smooth manifold of the same dimension.

(c) A Lie group is a C^∞-manifold G having a group structure such that the multiplication map

$$\mu : G \times G \to G$$

and the inverse map

$$\iota : G \to G, \ \iota(c) = x^{-1}$$

are both C^∞. Show that $GL(n, \mathbb{R})$ is a Lie group and compute its dimension.

Solution:

(a) Let $\{(U_i, \varphi_i)\}_{i \in \mathcal{I}}$, $\{(V_j, \psi_j)\}_{j \in \mathcal{J}}$ be an atlas of M_1 and M_2, respectively. We claim that $\{(U_i \times V_j, (\varphi_i, \psi_j))\}_{(i,j) \in \mathcal{I} \times \mathcal{J}}$ is an atlas for the product manifold, where $(\varphi_i, \psi_j)(p_1, p_2) := (\varphi_i(p_1), \psi_j(p_2)) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \cong \mathbb{R}^{m_1 + m_2}$.

Note that $M_1 \times M_2$ is second countable and Hausdorff since M_1 and M_2 are. Moreover, the (φ_i, ψ_j) are homeomorphisms since their components are and the sets $\{U_i \times V_j\}_{i \in \mathcal{I}, j \in \mathcal{J}}$ cover $M \times N$. Thus $M_1 \times M_2$ is a topological manifold of dimension $(m_1 + m_2)$. The transition functions

$$(\varphi_i, \psi_j) \circ (\varphi_k, \psi_l)^{-1} = (\varphi_i \circ \varphi_k^{-1}, \psi_j \circ \psi_l^{-1})$$

are smooth on $(\varphi_k, \psi_l)((U_i \times V_j) \cap (U_k \times V_l)) = \varphi_k(U_i \cap U_k) \times \psi_l(V_j \cap V_l)$ because M and N are smooth manifolds. Furthermore,

$$\varphi_i \circ \pi_1 \circ (\varphi_k, \psi_l)^{-1} = \varphi_i \circ \varphi_k^{-1}$$

is smooth on $\varphi_k(U_i \cap U_k) \times \psi_l(V_l)$, therefore π_1 is a smooth function and by a similar argument, the same follows for π_2.

(b) Let $\{(U_i, \varphi_i)\}_{i \in \mathcal{I}}$ be an atlas for the smooth manifold M and $V \subset M$ open. Then $\{(U_i \cap V, \varphi_i|_{U_i \cap V})\}_{i \in \mathcal{I}}$ is an atlas for V and the transition functions of the charts restricted to V are still smooth. This makes V into a smooth manifold of the same dimension as M.

(c) We identify the $n \times n$ matrices with $\mathbb{R}^{n^2}$, since both spaces are isomorphic. By definition,

$$GL(n, \mathbb{R}) = \{A \in \mathbb{R}^{n \times n} | \det(A) \neq 0\}.$$

Hence, for $\det : \mathbb{R}^{n \times n} \to \mathbb{R}$, we have that $GL(n, \mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$, which is open since det is continuous. By part (c) of this exercise, it is a smooth manifold of dimension n^2. Let A, $B \in \mathbb{R}^{n \times n}$. As

$$\mu(A, B)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj},$$

multiplication is smooth. By Cramer's rule,

$$\iota(A)_{ij} = \frac{1}{\det(A)} (-1)^{i+j} ((j, i)\text{−minor of } A).$$

As $\det(A) \neq 0$, inversion is also smooth and consequently $GL(n, \mathbb{R})$ is a Lie group.