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Differential Topology: Exercise Sheet 1

Exercises (for Oct. 24th and 25th)

1.1 Examples for quotient topologies

Draw the following topological spaces:

(a) Let $V := [-1,1] \times [0,1] \subset \mathbb{R}^2$ with the subspace topology and $\sim$ the equivalence relation such that

$$(x,0) \sim (x,1) \qquad \forall x \in [-1,1]$$

and all other (x,t), $0 < t < 1$ are only equivalent to themselves. Consider $V/\sim$.

(b) Let $W := [-1,1] \times [0,1] \subset \mathbb{R}^2$ with the subspace topology and $\sim$ the equivalence relation such that

$$(x,0) \sim (-x,1) \qquad \forall x \in [-1,1]$$

and all other (x,t), $0 < t < 1$ are only equivalent to themselves. Consider $W/\sim$.

(c) Define the group action $(\mathbb{Z} \times \mathbb{Z}) \times \mathbb{R}^2 \to \mathbb{R}^2$ as $(m,n) \cdot (x,y) \mapsto (x+m, y+n)$ for m, $n \in \mathbb{Z}$, x,$y \in \mathbb{R}$. Consider $\mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z})$. By this, we mean the quotient space of $\mathbb{R}^2$ with respect to the equivalence relation

$$(u,v) \sim (x,y) \text{ if } \exists (m,n) \in \mathbb{Z}\times\mathbb{Z} \text{ such that } (m,n)\cdot(u,v) = (x,y) \text{ for } (x,y),(u,v) \in \mathbb{R}^2\,.$$

(d) Identify $S^1 \simeq \{z \in \mathbb{C} |\ |z| = 1\}$. Let $Y := S^1 \times [0,1]$ and $\sim$ the equivalence relation such that

$$(z,0) \sim (\overline{z},1) \qquad \forall z \in \mathbb{C}$$

and all other (z,t), $0 < t < 1$ are only equivalent to themselves. Consider $Y/\sim$.

Solution:

(a) Cylinder
(b) Moebius strip
(c) Torus
(d) Klein bottle

1.2 Hausdorff property of quotient spaces

In this exercise you will show that the Hausdorff separation property of a given topological space does generally not extend to a quotient space thereof.

Consider $X := [-1,1] \times \{0,1\} \subset \mathbb{R}^2$ with the subspace topology (inherited from the natural topology on $\mathbb{R}^2$) that is Hausdorff and the equivalence relation

$$(x,0) \sim (x,1) \qquad \forall x > 0\,. \tag{1}$$

Show that the quotient space $X/\sim$ with the quotient topology is not Hausdorff.

Solution: Set $x = (0,0)$ and $y = (0,1)$. Consider the disjoint points $[x]$ and $[y]$ in the quotient space $X/\sim$. Let U_x and U_y be open neighborhoods of $[x]$ and $[y]$, respectively. Their pre-images (under the quotient map q) are open subsets of X and hence we can find open balls (the open sets in $\mathbb{R}^2$) centered around the points x and y with radii ϵ_x and ϵ_y, respectively, such that $B_{\epsilon_x}(x) \cap X \subset q^{-1}(U_x)$ and $B_{\epsilon_y}(y) \cap X \subset q^{-1}(U_y)$. Now choose $\epsilon > 0$ such that $\epsilon < \epsilon_x$ and $\epsilon < \epsilon_y$, then

$$(\epsilon, 0) \in B_{\epsilon_x}(x) \cap X \subset q^{-1}(U_x) \quad , \quad (\epsilon, 1) \in B_{\epsilon_y}(y) \cap X \subset q^{-1}(U_y) \ . \tag{2}$$

In $X/\sim$, the elements $[(\epsilon, 0)] \in U_x$ and $[(\epsilon, 1)] \in U_y$. Due to the equivalence relation, we have $(\epsilon, 0) \sim (\epsilon, 1)$ and therefore $[(\epsilon, 0)] = [(\epsilon, 1)] \in U_x \cap U_y$.

1.3 Another basic notion from topology

Show that a topological space X is connected if and only if every continuous $g : X \mapsto \{0,1\}$ is constant, where $\{0,1\}$ has the discrete topology. *Note that continuity of a map between two topological spaces will be treated in the lecture on Oct. 22nd.*

Solution: Let X be connected, assume that there is a continuous non-constant $g : X \mapsto \{0,1\}$. Then $U = g^{-1}(\{0\})$, $V = g^{-1}(\{1\})$ are open, disjoint and their union is X, which is a contradiction. Assume now that every continuous $g : X \mapsto \{0,1\}$ is constant and suppose that X is not connected. Then there are disjoint open $U, V \subset X$ such that their union is X and

$$g(x) = \begin{cases} 0 & x \in U \\ 1 & x \in V \end{cases}$$

is a continuous function, a contradiction.

1.4 Infinitude of prime numbers

In this exercise you will show that there are infinitely many prime numbers.
Consider the set $\mathbb{Z}$ and its affine subsets $S_{(a,b)} := \{an + b | n \in \mathbb{Z}\} = a\mathbb{Z} + b$ for $a, b \in \mathbb{Z}$ and $a \neq 0$. We define the open sets as those that are either empty or arbitrary unions or finite intersections of sets $S_{(a,b)} = \{an + b | n \in \mathbb{Z}\}$. Show that:

(a) This defines a topology on $\mathbb{Z}$;

(b) No non-empty finite subset of $\mathbb{Z}$ is open; *Hint: show that every intersection of two – or finitely many – affine subsets contains an (infinite) affine subset;*

(c) Each affine subset $S_{(a,b)} = \{an + b | n \in \mathbb{Z}\}$ is both open and closed;

(d) There are infinitely many prime numbers by analyzing the set

$$S := \bigcup_p S_{(p,0)} \qquad \text{where } p \text{ prime} \ , \tag{3}$$

and using the above properties (b) and (c) of the topology.

Solution:
This is a proof originally introduced by Fürstenberg.
(a) The empty set is in the topology by definition. The whole set can be written as $\mathbb{Z} = S_{(1,0)}$ and hence also is. Unions and finite intersections of sets $S_{(a,b)}$ are open by definition.
(b) Assume that the intersection of affine sets $S_{(a_i,b_i)}$ for $i = 1, \ldots, k$ is non-empty. Then

there is an $x \in S_{(a_i,b_i)}$ for all $i = 1, \ldots, k$. In other words, we can find $n_1, \ldots, n_k \in \mathbb{Z}$ such that $x = a_i n_i + b_i$ for all $i = 1, \ldots, k$. For some $n \in \mathbb{Z}$ let us define $y_n = n a_1 \cdots a_k$, i.e. it is a multiple of the a_i, then $x + y_n \in S_{(a_i,b_i)}$ for all $i = 1, \ldots, k$. Since this holds for all $n \in \mathbb{Z}$ we found an affine (therefore infinite) subset $S_{(a_1 \ldots a_k, x)}$ of all sets $S_{(a_i,b_i)}$ for $i = 1, \ldots, k$. As a consequence any (non-empty) open set contains an affine subset and hence is infinite. In other words, no non-empty finite subset of $\mathbb{Z}$ is open.
(c) Let us consider an affine set $S_{(a,b)}$ that is open by definition. Its complement is open since it can be written as the union of affine and hence open sets, i.e. as

$$\mathbb{Z} \setminus S_{(a,b)} = \bigcup_{j=1}^{a-1} S_{(a,b+j)} \ . \tag{4}$$

Hence $S_{(a,b)}$ is also closed.
(d) Assume that the number of primes is finite, i.e. there is an $m \in \mathbb{N}$ such that the prime numbers are $p_1, \ldots, p_m$. Then the set S would be a finite union of the sets $S_{(p_1,0)}, \ldots, S_{(p_m,0)}$. Consider the complement of S that can be written as

$$\mathbb{Z} \setminus S = \left(\mathbb{Z} \setminus S_{(p_1,0)}\right) \cap \cdots \cap \left(\mathbb{Z} \setminus S_{(p_m,0)}\right) \tag{5}$$

which is a finite intersection of the sets $\mathbb{Z} \setminus S_{(p_1,0)}, \cdots, \mathbb{Z} \setminus S_{(p_m,0)}$. Since the affine sets $S_{(p_1,0)}, \ldots, S_{(p_m,0)}$ are closed due to property (c), their complements are open. Hence $\mathbb{Z} \setminus S$ is open as the finite intersection of open sets. Its complement S is therefore closed.
On the other hand, the complement of S is the set $\{-1, 1\}$ (any other integer can be written as $n \cdot p$ for $n \in \mathbb{Z}$ and p prime due to prime factoring). This is a finite non-empty subset of $\mathbb{Z}$. Property (b) hence implies that it cannot be open. Therefore its complement, the set $S = \mathbb{Z} \setminus \{-1, 1\}$, cannot be closed which is a contradiction.