PROF. DR. M. WOLF
M. HEINZE

Differential Topology: Exercise Sheet 4

Exercises (for Dec. 19th and 20th)

4.1 Application 1 of Brouwer's fixed point theorem: Perron-Frobenius

In the following, we use the continuous version of Brouwer's fixed-point theorem: every continuous function $f : D^n \to D^n$ has a fixed point where D^n is the closed unit ball in $\mathbb{R}^n$. In this exercise, we show the Perron-Frobenius theorem:

> A matrix $A \in \mathbb{R}^{n\times n}$ with $A_{i,j} \geq 0$ for all $i, j \in \{1, \ldots, n\}$ has a non-negative eigenvalue.

In the following, think of A as a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ (expressed in the standard basis).

(a) Show that

$$S_+^{n-1} = \{x = (x_1, \ldots, x_n) \in S^{n-1} \mid x_j \geq 0 \text{ for all } j = 1, \ldots, n\} \tag{1}$$

is homeomorphic to D^{n-1}.

(b) Consider the map $f(x) = Ax/\|Ax\|$ to prove the theorem.

4.2 Counterexample to Brouwer's fixed point theorem in infinite dimensions

In this exercise, we show that Brouwer's fixed point theorem does not extend to infinite dimensions, i.e. in an infinite-dimensional space, not every continuous function $f : D \to D$ (where D is the closed unit ball in the infinite-dimensional space) to the closed unit ball has a fixed point.

For example, consider the space ℓ_2 of square summable sequences, i.e., $x = (x_1, x_2, \ldots) \in \ell_2$ if and only if $x_i \in \mathbb{R}$ and $\|x\|_2^2 = \sum_{i=1}^{\infty} x_i^2 < \infty$ and let $D = \{x \in \ell_2 \mid \|x\|_2^2 \leq 1\}$ denote the closed unit ball in ℓ_2. Use this space to construct a counter example: find a continuous function $f : D \to D$ that has no fixed point.

4.3 Classification of compact 1-manifolds with smooth embedding

Let M be a 1-dimensional compact connected manifold with boundary and assume that there is a smooth embedding into some $\mathbb{R}^n$. Show that there must be two boundary points.

Hint: Starting from one boundary point, parametrize the manifold in terms of its arc length and show that the arc length is finite.