Technische Universität München

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Sheet 4

Differential Topology: Exercise Sheet 4

Exercises (for Dec. 5th and 6th)

4.1 Different definitions of tangent space

Consider the geometrically more intuitive definition of a tangent space from the motivating example in the lecture: Let M be a smooth submanifold embedded in some $\mathbb{R}^n$ and define

$$\vec{T}_x M := \{v \in \mathbb{R} \mid \gamma \in C^\infty\big((-1,1), M\big), \gamma(0) = x, \gamma'(0) = v\} \qquad (1)$$

Show that the the vector spaces $T_x M$ and $\vec{T}_x M$ are isomorphic, i.e, that the map $T_x M \to \vec{T}_x M$, $[\gamma] \mapsto \gamma'(0)$ is a vector space isomorphism.

4.2 Lie group actions

Let G denote a group and X an arbitrary set. A (left) group action of G on X is a map

$$\alpha : G \times X \to X$$

which has the properties

- $\alpha(e, \cdot) = \mathrm{id}_X$ for the unit element $e \in G$
- $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$ for all $g, h \in G$, $x \in X$.

In the case where G is a Lie group and X is a smooth manifold, we call a Lie group action a smooth group action of G on X, i.e. α is a smooth map such that $\alpha(g, \cdot) : X \to X$ is a diffeomorphism for every $g \in G$. In the following let α denote a Lie group action of the Lie group G on the smooth manifold X.

(a) Show that $\alpha(\cdot, x) : G \to X$ has constant rank for all $x \in X$. (Hint: Use that the left-multiplication $L_g : G \to G$ $L_g(h) = gh$ is a diffeomorphism.)

(b) Let $G_x = \{h \in G | \alpha(h, x) = x\}$ be called the stabilizer of $x \in X$ and let $U \cdot G_x = \{g \cdot h | g \in U, h \in G_x\}$. Show that $\alpha(U \cdot G_x, x) = \alpha(U, x)$ and that for $U \subset G$ open, $U \cdot G_x$ is open.

(c) Let G be a compact Lie group, $U \subset G$ open and $x \in X$. Show that $\alpha(U, x)$ is open.

(d) Let G be a compact Lie group. Show that the orbit of $x \in X$ under the action α of G on X, defined as

$$\mathcal{O}_x = \alpha(G, x),$$

is a smooth submanifold of X.

(e) Consider the special case $G = U(n)$ of $n \times n$ unitary matrices and $X = \mathcal{H}_n$ of $n \times n$ Hermitian matrices. Show that the map $\alpha : U(n) \times \mathcal{H}_n \to \mathcal{H}_n$ defined by $\alpha(U, A) = UAU^*$ is a Lie group action.

(f) Show that the unitary equivalence orbit of $A \in \mathcal{H}_n$, denoted by $\mathcal{O}_A = \{UAU^* | U \in U(n)\}$ is a smooth manifold.