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Sheet 3

Differential Topology: Exercise Sheet 3

Exercises (for Nov. 21th and 22th)

3.1 Smooth maps

Prove the following:

(a) A map $f : M \to N$ between smooth manifolds $(M, \mathcal{A})$, $(N, \mathcal{B})$ is smooth if and only if for all $x \in M$ there are pairs $(U, \phi) \in \mathcal{A}$, $(V, \psi) \in \mathcal{B}$ such that $x \in U$, $f(U) \subset V$ and $\psi \circ f \circ \phi^{-1}; \phi(U) \to \psi(V)$ is smooth.

(b) Compositions of smooth maps between subsets of smooth manifolds are smooth.

3.2 Mazur's swindle

The connected sum $\sharp$ is a basic operation on oriented, connected, compact, n-dimensional manifolds. It has a number of interesting properties. One can show that

(a) $M \sharp S^n \simeq M$ (unit element)

(b) $(M \sharp N) \sharp P \simeq M \sharp (N \sharp P)$ (associativity)

(c) $M \sharp N \simeq N \sharp M$ (commutativity)

for n-dimensional manifolds M, N, P where $\simeq$ denotes equal up to homeomorphisms.

Show that the sphere S^n is itself irreducible, i.e. if $S^n \simeq M \sharp N$ for n-dimensional manifolds M, N, then $M, N \simeq S^n$.

Note: You can use the above properties without proof. Note that the associativity also holds for a connected sum of infinitely many topological manifolds.

3.3 System of inequalities

Is the set $S := \{x \in \mathbb{R}^3 \mid \sum_{i=1}^3 x_i^3 = 1, \text{ and } \sum_{i=1}^3 x_i = 0\}$ a smooth submanifold of $\mathbb{R}^3$?

3.4 Lie groups

(a) Let G be a Lie group and $H \subset G$ a smooth submanifold that is also a subgroup of G. Show that H is a Lie group as well.

(b) Define the block matrix

$$\sigma := \bigoplus_{k=1}^{n} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

and the **real symplectic group** $\mathsf{Sp}(2n, \mathbb{R}) := \{S \in \mathbb{R}^{2n \times 2n} \mid S \sigma S^T = \sigma\}$. Prove that $\mathsf{Sp}(2n, \mathbb{R})$ with the matrix multiplication and the matrix inversion forms a Lie group. What is the manifold dimension of $\mathsf{Sp}(2n, \mathbb{R})$?

3.5 Immersions and embeddings

(a) Formalize and prove the statement: an immersion is locally an embedding.

(b) Let $(M, \mathcal{A})$, $(N, \mathcal{B})$ denote two smooth manifolds. Show that $f : M \to N$ is an embedding if and only if $f : M \to f(M)$ is a diffeomorphism.