Technische Universität München

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Sheet 1

Differential Topology: Exercise Sheet 1

Exercises (for Oct. 24th and 25th)

1.1 Examples for quotient topologies

Draw the following topological spaces:

(a) Let $V := [-1,1] \times [0,1] \subset \mathbb{R}^2$ with the subspace topology and $\sim$ the equivalence relation such that

$$(x,0) \sim (x,1) \qquad \forall x \in [-1,1]$$

and all other (x,t), $0 < t < 1$ are only equivalent to themselves. Consider $V/\sim$.

(b) Let $W := [-1,1] \times [0,1] \subset \mathbb{R}^2$ with the subspace topology and $\sim$ the equivalence relation such that

$$(x,0) \sim (-x,1) \qquad \forall x \in [-1,1]$$

and all other (x,t), $0 < t < 1$ are only equivalent to themselves. Consider $W/\sim$.

(c) Define the group action $(\mathbb{Z} \times \mathbb{Z}) \times \mathbb{R}^2 \to \mathbb{R}^2$ as $(m,n) \cdot (x,y) \mapsto (x+m, y+n)$ for m, $n \in \mathbb{Z}$, x,$y \in \mathbb{R}$. Consider $\mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z})$. By this, we mean the quotient space of $\mathbb{R}^2$ with respect to the equivalence relation

$$(u,v) \sim (x,y) \text{ if } \exists (m,n) \in \mathbb{Z}\times\mathbb{Z} \text{ such that } (m,n)\cdot(u,v) = (x,y) \text{ for } (x,y),(u,v) \in \mathbb{R}^2 .$$

(d) Identify $S^1 \simeq \{z \in \mathbb{C} |\ |z| = 1\}$. Let $Y := S^1 \times [0,1]$ and $\sim$ the equivalence relation such that

$$(z,0) \sim (\overline{z},1) \qquad \forall z \in \mathbb{C}$$

and all other (z,t), $0 < t < 1$ are only equivalent to themselves. Consider $Y/\sim$.

1.2 Hausdorff property of quotient topologies

In this exercise you will show that the Hausdorff separation property of a given topology does generally not extend to quotient topologies thereof.

Consider $X := [-1,1] \times \{0,1\} \subset \mathbb{R}^2$ with the subspace topology (inherited from the natural topology on $\mathbb{R}^2$) that is Hausdorff and the equivalence relation

$$(x,0) \sim (x,1) \qquad \forall x > 0 . \tag{1}$$

Show that the quotient topology $X/\sim$ is not Hausdorff.

1.3 Another basic notion from topology

Show that a topological space X is connected if and only if every continuous $g : X \mapsto \{0,1\}$ is constant, where $\{0,1\}$ has the discrete topology. *Note that continuity of a map between two topological spaces will be treated in the lecture on Oct. 22nd.*

1.4 Infinitude of prime numbers

In this exercise you will show that there are infinitely many prime numbers.
Consider the topology on $\mathbb{Z}$ generated by affine subsets $S_{(a,b)} := \{an + b | n \in \mathbb{Z}\} = a\mathbb{Z} + b$ for $a \neq 0$, i.e. open sets are those that are either empty or arbitrary unions or finite intersections of sets $S_{(a,b)} = \{an + b | n \in \mathbb{Z}\}$. Show that:

(a) This defines a topology on $\mathbb{Z}$;

(b) No non-empty finite subset of $\mathbb{Z}$ is open; *Hint: show that every intersection of two – or finitely many – affine subsets contains an (infinite) affine subset;*

(c) Each affine subset $S_{(a,b)} = \{an + b | n \in \mathbb{Z}\}$ is both open and closed;

(d) There are infinitely many prime numbers by analyzing the set

$$S := \bigcup_p S_{(p,0)} \qquad \text{where } p \text{ prime} \quad , \tag{2}$$

and using the above properties (b) and (c) of the topology.