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Differential Topology: Exercise Sheet 7

Exercises (for Jan. 30th and 31th)

7.1 Smooth homotopy

Two smooth mappings $f, g : M \to N$ (for M, N smooth manifolds) are called smoothly homotopic if there exists a smooth map $F : X \times [0, 1] \to Y$ with

$$F(x, 0) = f(x) \qquad F(x, 1) = g(x) \tag{1}$$

for all $x \in M$. Show that the relation of smooth homotopy is an equivalence relation.

7.2 Stack of records theorem

Let $f : M \to N$ denote a smooth map between the smooth compact manifold $(M, \mathcal{A})$ and the smooth manifold $(N, \mathcal{B})$ with $\dim(N) = \dim(M)$. Suppose $y \in N$ is a regular value of f.

(a) Show that $f^{-1}(y) \subset M$ is a finite set.

(b) For $k \in N$ assume $f^{-1}(y) = \{x_1, \ldots, x_k\}$ for $x_i \in M$. Prove that there exist a neighborhood $U \subset N$ of y and disjoint open neighborhoods $V_i \subset M$ of x_i for all $i \in \{1, \ldots, k\}$ such that $f^{-1}(U) = V_1 \sqcup \cdots \sqcup V_k$ and such that $f|_{V_i} : V_i \to U$ is a diffeomorphism for all $i \in \{1, \ldots, k\}$.

7.3 Boundary theorem

Let M, N be smooth manifolds with $\dim M = \dim N$. Suppose that $M = \partial W$ for a compact manifold W and let $g : M \to N$ be a smooth map. Show that if g extends to a smooth map $G : W \to N$ then $\deg_2(g) = 0$.

Hint: You may use the fact that compact, one-dimensional manifolds have an even number of boundary points.

7.4 No-retraction theorem

Let W be a compact manifold with non-empty connected boundary $\partial W \neq \emptyset$. Show that there is no retraction of W to ∂W, i.e. there is no smooth map $f : W \to \partial W$ such that $f|_{\partial W} = \mathsf{id}_{\partial W}$.

Hint: Use the boundary theorem from the previous exercise.