Prof. Dr. M. Wolf
M. Heinze

Differential Topology: Exercise Sheet 2

Exercises (for Nov. 7th and 8th)

2.1 Real projective space

Consider the equivalence relation

$$x \sim y \qquad :\Leftrightarrow \qquad \exists \lambda \in \mathbb{R}\backslash\{0\} : x = \lambda y \tag{1}$$

on $\mathbb{R}^{n+1}\backslash\{0\}$. Then $\mathbb{R}P^n := (\mathbb{R}^{n+1}\backslash\{0\})/\sim$ is called the **(real) projective space.**

(a) Show that $\mathbb{R}P^n$ is homeomorphic to $S^n/\sim$ where the relative topology and the equivalence relation

$$x \sim y :\Leftrightarrow x = \pm y \tag{2}$$

on the unit sphere $S^n \subset \mathbb{R}^{n+1}$ are used. This provides an alternative definition of $\mathbb{R}P^n$.

In the following, we show that $\mathbb{R}P^n$ is a smooth n-manifold:

(b) Show that $\mathbb{R}P^n$ is Hausdorff.

(c) Show that $\mathbb{R}P^n$ is second countable.

(d) Define $U_j := \{x \,|x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}, x_j \neq 0\}$ and let $q : \mathbb{R}\backslash\{0\} \to \mathbb{R}P^n$ be the quotient map associated to the equivalence relation (1). Show that $\{q(U_j)\}_{j=1}^n$ is an open cover of $\mathbb{R}P^n$, and that each $q(U_j)$ is homeomorphic to a subset of $\mathbb{R}^n$.

(e) Show that $\mathbb{R}P^n$ is a smooth manifold. Show that the quotient map $q : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}P^n$ used to define $\mathbb{R}P^n$ is smooth.

2.2 Boundary and interior of a topological manifold

Let M be an n-dimensional topological manifold with non-trivial boundary ∂M. Show that

(a) $\mathsf{int}(M)$ is an n-dimensional manifold,

(b) ∂M is an $(n-1)$-dimensional manifold.

2.3 Examples of differentiable manifolds

(a) Suppose M_j are smooth m_j-manifolds, for $j = 1, 2$. Show that there is a smooth structure on $M_1 \times M_2$ such that $M_1 \times M_2$ is a smooth $(m_1 + m_2)$-manifold, and the canonical projections $\pi_j : M_1 \times M_2 \to M_j$ are smooth, for $j = 1, 2$.

(b) Show that any open subset of a smooth manifold is again a smooth manifold of the same dimension.

(c) A Lie group is a C^∞-manifold G having a group structure such that the multiplication map

$$\mu : G \times G \to G$$

and the inverse map

$$\iota : G \to G, \;\; \iota(c) = x^{-1}$$

are both C^∞. Show that $GL(n, \mathbb{R})$ is a Lie group and compute its dimension.