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Differential Topology

http://www-m5.ma.tum.de/Allgemeines/MA5122_2014S
Sommers. 2014
Lösungsblatt 5
(04.06.2014)

Tutoraufgaben

1. A local Lemma

Let M be an m-dimensional smooth submanifold of an n-dimensional smooth manifold N.

Show that for any $x \in M$ there is a neighborhood $U \subset N$ of x and a smooth function $g : U \to \mathbb{R}^{n-m}$ such that 0 is a regular value of g and

$$M \cap U = g^{-1}(0).$$

LÖSUNG:

Let $\mathcal{A}$ denote the atlas of M. Since $M \subset N$ is a smooth submanifold, there is a chart $(U, \phi) \in \mathcal{A}$ such that $x \in U$ and with the inclusion $i(x_1, \ldots, x_m) := (x_1, \ldots, x_m, 0, \ldots, 0)$ we have $M \cap U = \phi^{-1}(i(\mathbb{R}^m))$.

Now we define $g : U \to \mathbb{R}^{n-m}$ such that $g \circ \phi^{-1}(x_1, \ldots, x_n) = (x_{m+1}, \ldots, x_n)$. Clearly g is smooth and a submersion, so 0 is a regular value and $g^{-1}(0) = \phi^{-1}(x_1, \ldots, x_m, 0, \ldots, 0) = M \cap U$.

2. Definition(Transversality)

Let N be a smooth manifold and $Z \subset N$ a smooth submanifold.

- Another submanifold $M \subset N$ is called **transverse** to Z (write $M \pitchfork Z$) if

$$T_zM + T_zZ = T_zN$$

holds for all $z \in M \cap Z$.

- A smooth map $f : X \to N$ is called **transverse** to Z if

$$df_x(T_xM) + T_{f(x)}Z = T_{f(x)}N$$

holds for all $x \in f^{-1}(Z)$ (so in particular if $f^{-1}(Z) = \emptyset$).

Note: The second definition includes the first, as the inclusion map $i : M \to N$ for a smooth submanifold $M \subset N$ is transversal to the submanifold $Z \subset N$ iff $M \pitchfork Z$.

LÖSUNG:

3. Transversality

Let $f : M \to N$ be a smooth map and $Z \subset N$ is a smooth submanifold with $f \pitchfork Z$ and $f^{-1}(Z) \neq \emptyset$.

(a) Show that $f^{-1}(Z)$ is a smooth submanifold of M.

(b) What is the dimension of this submanifold?

(c) Prove that $T_xf^{-1}(Z) = (df_x)^{-1}\left(T_{f(x)}Z\right)$

LÖSUNG:

(a) Consider $z := f(x) \in Z$. By the previous Lemma there is a neighborhood U and a smooth map $g : U \to \mathbb{R}^{\dim(N)-\dim(Z)}$ with $g^{-1}(0) = Z \cap U$ and such that 0 is a regular value of g, i.e. dg_z is surjective. Therefore $f^{-1}(U)$ is a neighborhood of x in M such that $f^{-1}(Z) \cap f^{-1}(U) = f^{-1}(Z \cap U) = f^{-1} \circ g^{-1}(0) = (g \circ f)^{-1}(0)$.
We want to show that 0 is a regular value of $g \circ f$, which then implies that $f^{-1}(Z)$ is a smooth submanifold. To this end we use transversality and get

$$df_x T_x M + T_z Z = T_z N$$

and by applying dg_z

$$d(g \circ f)_x T_x M + dg_z T_z Z = dg_z T_z N.$$

Now note that $dg_z T_z Z = 0$ since $\ker(dg_z) = T_z Z$ by the lecture and $dg_z T_z N = \mathbb{R}^{\dim(N)-\dim(Z)}$ since dg_z is surjective. Therefore $d(g \circ f)_x T_x M = \mathbb{R}^{\dim(N)-\dim(Z)}$, which implies that $d(g \circ f)_x$ is surjective and 0 is a regular value of $g \circ f$.

(b) By the regular value theorem and the previous exercise we have

$$\dim(f^{-1}(Z)) = \dim(M) - \dim(N) + \dim(Z).$$

(c) For the tangent space we obtain

$$T_x f^{-1}(z) = \ker(d(g \circ f)_x) = \ker(dg_z \circ df_x) = (df_x)^{-1}(\ker(dg_z)) = (df_x)^{-1}(T_z Z).$$

4. Transversal intersections

Let M and Z be two transverse smooth submanifolds of N. Prove that $M \cap Z$ is again a smooth submanifold of N with

$$\dim(M \cap Z) = \dim(M) + \dim(Z) - \dim(N).$$

LÖSUNG:

Applying the previous theorem to the inclusion map finishes the proof.

Hausaufgaben

5.1. Stack of Records Theorem

Let $f : M \to N$ denote a smooth map between the smooth compact manifold $(M, \mathcal{A})$ and the smooth manifold $(N, \mathcal{B})$ with $\dim(N) = \dim(M)$. Suppose $y \in N$ is a regular value of f.

(a) Show that $f^{-1}(y) \subset M$ is a finite set.

(b) For $k \in \mathbb{N}$ assume $f^{-1}(y) = \{x_1, \ldots, x_k\}$ for $x_i \in M$. Prove that there exist a neighborhood $U \subset N$ of y and disjoint open neighborhoods $V_i \subset M$ of x_i for all $i \in \{1, \ldots, k\}$ such that $f^{-1}(U) = V_1 \sqcup \ldots \sqcup V_k$ and such that $f|_{V_i} : V_i \to U$ is a diffeomorphism for all $i \in \{1, \ldots, k\}$.

LÖSUNG:

(a) Consider $x \in f^{-1}(y)$ and note that $\dim(M) = \dim(N)$ implies that the surjective map $df_x : T_x(M) \to T_y(N)$ is also injective as a linear map between vector spaces of the same dimension. Therefore by the inverse function theorem there exists an open neighborhood $V_x \subset M$ of $x \in M$ such that $f|_{V_x} : V_x \to f(V_x)$ is a diffeomorphism. This shows that $V_x \cap f^{-1}(y) = \{x\}$. As $f^{-1}(y)$ is closed (as a preimage of a continuous map of a closed set) and a subset of the compact manifold $(M, \mathcal{A})$ it is also compact. Therefore any cover $\cup_{x \in f^{-1}(y)} V_x$ with open sets V_x has a finite subcover, which shows that there are only finitely many points in $f^{-1}(y)$, because otherwise there exists a set V_x containing two distinct points $x, x' \in f^{-1}(y)$ which is a contradiction.

(b) Assume that there are $k \in \mathbb{N}$ points in $f^{-1}(y)$. Consider the open sets $\{V_{x_1}, \ldots, V_{x_k}\}$ defined in part (a) and shrink each of them until $V_{x_i} \cap V_{x_j} = \emptyset$ for all $i, j \in \{1, \ldots, k\}$. As the V_{x_i} are open sets $M \setminus (\cup_i V_{x_i})$ is a closed subset of a compact manifold M and therefore also compact. Using continuity of f this shows, that $f(M \setminus (\cup_i V_{x_i}))$ is also compact and in particular closed. By definition $y \notin f(M \setminus (\cup_i V_{x_i}))$ and thus there exists an open neighborhood $U \subset N$ of $y \in N$ such that $U \cap f(M \setminus (\cup_i V_{x_i})) = \emptyset$. This implies $f^{-1}(U) \subset \cup_i V_{x_i}$, which finishes the proof by taking sets $V_i = f^{-1}(U) \cap V_{x_i}$. With this definition we have $\sqcup_i V_i = f^{-1}(U) \cap \cup_i V_{x_i} = f^{-1}(U)$. Clearly $f|_{V_i}$ is a diffeomorphism as $V_i \subset V_{x_i}$ and also $V_i \cap V_j = \emptyset$ for all $i, j \in \{1, \ldots, k\}$.

5.2. Morse Functions

Let $(M, \mathcal{A})$ denote a smooth $m-$dimensional manifold and consider a smooth function $f : M \to \mathbb{R}$. By definition $a \in M$ is a critical point of f if there is a chart $(U, \phi) \subset \mathcal{A}$ on a neighborhood $U \ni a$ such that $\mathrm{dg}_{\phi(a)} = 0$ for the function $g = f \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}$.

The critical point $a \in M$ is called non-degenerate iff the Hessian $H \in \mathfrak{M}_m$ defined by $H_{ij} = \frac{\partial^2 g}{\partial x_i \partial x_j}(\phi(a))$ is non-singular.

(a) Show that the definition of non-degeneracy is well-defined, i.e. does not depend on the choice of charts.

(b) Let $U \subset \mathbb{R}^n$ denote an open set and $f : U \to \mathbb{R}$ a smooth function. Show that for almost all $a = (a_1, \ldots, a_n) \subset \mathbb{R}^n$ the function

$$f_a = f + a_1 x_1 + \ldots + a_n x_n$$

is a Morse function on U, i.e. all its critical values are non-degenerate.

(c) Prove that on every smooth n-manifold M there exist a Morse function $f : M \to \mathbb{R}$. (**Hint:** Choose an embedding of M into some $\mathbb{R}^N$ and use that M has a countable, locally finite cover (see Tutor exercise 2 on sheet 2)).

Note: This provides the missing piece in the proof of the classification theorem of compact 1-manifolds from sheet 3.

LÖSUNG:

(a) Consider a critical point $a \in M$ and assume that it is non-degenerate wrt the chart (U_1, ϕ_1) on an open neighborhood $U_1 \in M$ of a. Now consider a different chart (U_2, ϕ_2) on an open neighborhood $U_2 \in M$ of a. We have $f \circ \phi_2^{-1} = f \circ \phi_1^{-1} \circ \psi$ for the diffeomorphism $\psi = \phi_1 \circ \phi_2^{-1} : \phi_2(U_1 \cap U_2) \to \phi_1(U_1 \cap U_2)$. As we can always add constants without changing the derivatives of the functions in question, we can assume wlog that $\phi_1(a) = \phi_2(a) = 0$. Therefore it is sufficient to show, that for an arbitrary function $g : \mathbb{R}^n \to \mathbb{R}$ with a non-degenerate critical point at 0, and for an arbitrary

diffeomorphism $\psi : \mathbb{R}^n \supset U \to V \subset \mathbb{R}^n$ with $\psi(0) = 0$ the function $g' = g \circ \psi : U \to \mathbb{R}$ has also a non-degenerate critical point at 0.
Using the chain rule we can compute

$$\frac{\partial g'}{\partial x_i}(x) = \sum_{\alpha=1}^{n} \frac{\partial g}{\partial x_\alpha}(\psi(x)) \frac{\partial \psi_\alpha}{\partial x_i}(x)$$

and furthermore

$$\frac{\partial^2 g'}{\partial x_i \partial x_j}(0) = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \frac{\partial^2 g}{\partial x_\alpha \partial x_\beta}(0) \frac{\partial \psi_\alpha}{\partial x_i}(0) \frac{\partial \psi_\beta}{\partial x_j}(0) + \sum_{\alpha=1}^{n} \frac{\partial g}{\partial x_\alpha}(0) \frac{\partial^2 \psi_\alpha}{\partial x_i \partial x_j}(0) \quad .$$

The second sum vanishes as 0 is a critical point of g and writing the above more compactely we get

$$H_0' = (d\psi_0)^T H_0 (d\psi_0) \quad ,$$

for the Hessians H_0', H_0 of g', g at 0. Because ψ is a diffeomorphism we know that $d\psi_0$ is invertible and we immediately get that H_0' is non-singular iff H_0 is non-singular.

(b) Define the function $g : U \to \mathbb{R}^n$ by $g := \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$. Consider now the function $f_a : U \to \mathbb{R}$ defined as above and compute

$$(df_a)_x = \left(\frac{\partial f_a}{\partial x_1}(x), \ldots, \frac{\partial f_a}{\partial x_n}(x)\right) = g(x) + a \quad .$$

Thus $x \in U$ is a critical point of f_a iff $g(x) = -a$ and it is easy to see that the Hessian H_x of f at a point $x \in U$ can be written as $H = dg_x$. Therefore H_x is non-singular if $g(x) = -a$ for a regular value $-a$ of g, which shows that all critical values of f_a are non-degenerate if $-a$ is a regular value of g. But as $g : U \to \mathbb{R}^n$ is a smooth map between smooth manifolds, almost every $-a \in \mathbb{R}^n$ is a regular value by Sard's theorem. This finishes the proof.

(c) Consider an embedding $e : M \to \mathbb{R}^N$ for some $N \in \mathbb{N}$ and denote by $(x_1, \ldots, x_N)$ the standard coordinate system on $\mathbb{R}^N$. We can cover $e(M)$ with countably many open sets $\{U_\alpha\}$ such that for each U_α the standard coordinate system can be restricted to a local coordinate system $(x_{\alpha_1}, \ldots, x_{\alpha_n})$. This follows from the constant rank theorem saying that for each $x \in M$ there exist two charts (W, ϕ) with $\mathbb{R}^N \supseteq W \ni e(x)$ and (V, ψ) with $\mathbb{R}^{\dim(M)} \supseteq V \ni x$ such that $\phi \circ e \circ \psi^{-1}(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0)$. As ϕ is a diffeomorphism this says that $U = e(V)$ has a local coordinate system, which is a restriction of the standard coordinate system to m components. As M is second-countable it is enough to consider countably many U_α to cover M.
On U_α define the function $f : U_\alpha \to \mathbb{R}$ as

$$f(x) = a_{n+1} x_{\alpha_{n+1}} + \cdots + a_N x_{\alpha_N}$$

for fixed arbitrary $(a_{n+1}, \ldots, a_N) \in \mathbb{R}^{N-n}$. Now the previous exercise shows, that for almost all $(a_1, \ldots, a_n) \in \mathbb{R}^n$ the function $f_a : U_\alpha \to \mathbb{R}$ defined as

$$f_a(x) = f(x) + a_1 x_{\alpha_1} + \ldots + a_n x_{\alpha_n}$$

is a Morse function. Note that we chose $\{U_\alpha\}$ in such a way, that the above function can be trivially extended to $f_a : e(M) \to \mathbb{R}$ by using the standard coordinate system on $\mathbb{R}^N$. The proof now follows by considering the sets

$$A_\alpha = \{a \in \mathbb{R}^N | f_a \text{ is not a Morse function on } U_\alpha\}.$$

By the previous exercise we know that A_α is of measure zero for all α, which implies that

$$\bigcup_{\alpha\in\mathbb{N}} A_\alpha$$

has measure also zero. Therefore f_a is a Morse function for almost all $a \in \mathbb{R}^N$. By the construction of the cover U_α it follows that $F_a := f_a \circ e$ is a Morse function on M for almost every $a \in \mathbb{R}$ (consider the charts ψ from above!).

5.3. Perron-Frobenius Theorem

Let $A \in \mathfrak{M}_n$ denote an $n \times n-$matrix with entries $a_{ij} > 0$ for all $1 \leq i, j \leq n$.

Show that there exists an eigenvector $v \in \mathbb{R}^n_+$ with entries $v_i \geq 0$ for all $1 \leq i \leq n$ with a corresponding eigenvalue $\lambda > 0$.

Hint: Consider a function of the form $f(x) = \frac{Ax}{\|Ax\|}$ acting on a convenient subset of $\mathbb{R}^n$.

LÖSUNG:

Consider the set $S = \{x \in \mathbb{R}^n | \forall i \in [n] : x_i \geq 0, \|x\|_2 = 1\}$, where $\| \bullet \|_2 : \mathbb{R}^n \to \mathbb{R}_+$ denotes the usual euclidean 2-norm. Now we define the continuous function $f : S \to S$ via $f(x) = \frac{Ax}{\|Ax\|_2}$. If $S \simeq D^{n-1}$ we are done, as Brouwer's fixpoint theorem would yield a fixpoint $v \in S$ such that $f(v) = v$, i.e. $Av = \|Av\|_2 v$. As $0 \notin S$ we found the eigenvector $v \in \mathbb{R}^n_+$ corresponding to the eigenvalue $\lambda = \|Av\|_2 > 0$. It is left to show, that $S \simeq D^{n-1}$, which we will only sketch here. Note that by our choice of the euclidean 2-norm, $S \subset S^{n-1}$ and it is exactly the part of the sphere lying in the first quadrant of the coordinate system (if we embed S^{n-1} into $\mathbb{R}^n$). By a rotation of this section around the first coordinate axis by n times the angle of the section, we obtain a homeomorphism mapping from the section to the half-sphere S^{n-1}_+ where only the first coordinate has to be non-negative. Projecting the half-sphere down to $\mathbb{R}^{n-1}$ yields a homeomorphism to D^{n-1}, which finishes the proof.

5.4. Tangent spaces of transversal intersections

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be transversal submanifolds of $(Z, \mathcal{C})$. Show that for $x \in X \cap Y$, we have

$$T_x(X \cap Y) = T_x(X) \cap T_x(Y) \quad .$$

LÖSUNG:

Consider the inclusion map $i_X : X \to Z$, defined by $i_X(x) = x \in Z$. The smooth submanifolds X and Y are transversal iff the inclusion map i_X is transversal to the manifold Y. According to a theorem from the lecture this implies that $i_X^{-1}(Y) = X \cap Y$ is a smooth submanifold of Z, and also according to the theorem we have

$$T_x(X \cap Y) = T_x\left(i_X^{-1}(Y)\right) = (di_X)_x^{-1}(T_x(Y))$$

for all $x \in X \cap Y$. But $(di_X)_x : T_x(X) \to T_x(Z)$ is just the inclusion map of $T_x(X)$ into $T_x(Z)$. Therefore we have

$$T_x(X \cap Y) = (di_X)_x^{-1}(T_x(Y)) = T_x(X) \cap T_x(Y)$$

for all $x \in X \cap Y$.