Corollary: (Whitney's embedding theorem – easy version)

If M is a compact manifold of dim. m, then it can be embedded in $\mathbb{R}^{2m+1}$.

proof: By a smooth version of the embedding thm. we can find an embedding into some $\mathbb{R}^N$. Due to compactness injective immersions are embeddings. Iterating the above thm. we can reduce the dimension down to $2m+1$. $\square$

remarks: We can learn two more things from the proof of the thm.:

- embeddings are not exceptional and, in fact, dense in the set of all smooth maps into $\mathbb{R}^{2m+1}$ (see [Hirsch] for a proof).
- if we are only interested in an immersion we can reduce the dimension further by one via restricting ourselves to unitvectors in the tangent space

Thm.: (Whitney's embedding & immersion theorem – strong version)

Any smooth m-dimensional manifold can be embedded in $\mathbb{R}^{2m}$ and immersed in $\mathbb{R}^{2m-1}$ if $m > 1$.

remarks:

- this is the best possible linear bound since $\mathbb{RP}^m$ cannot be embedded in $\mathbb{R}^{2m-1}$ if m is a power of two.
- for immersions a tight bound is known (proven by Cohen in '85): any m-dim smooth compact manifold can be immersed in $\mathbb{R}^{2m-\alpha(m)}$, where $\alpha(m)$ is the number of ones in the binary expansion of m.

X. Homotopy & mod 2-degree

"Homotopy" formalizes continuous (or smooth) deformations of maps.

Def.: • Let $f, g: M \to N$ be smooth maps between smooth manifolds. f & g are called "smoothly homotopic" if there is a smooth map $F: M \times [0,1] \to N$ such that $\forall x \in M$: $F(x,0) = f(x) \wedge F(x,1) = g(x)$.
We write $F_t(x) := F(x,t)$.

• F is then called a "smooth homotopy" between f & g.

• The equivalence class $[f]$ of maps smoothly homotopic to f is its "smooth homotopy class".

Recall the "stack of records thm.": M compact, $f: M \to N$ smooth, $\dim M = \dim N$, $y \in N$ regular value (i.e. $f(x) = y \Rightarrow df_x$ surjective). Then

(i) $f^{-1}(y) = \{x_1, \dots, x_k\}$ is finite

(ii) $\exists$ open neighborhood $U \ni y$ s.t. $f^{-1}(U) = V_1 \sqcup \dots \sqcup V_k$ where each V_j is an open neighborhood of x_j and $f: V_j \to U$ is a diff.

$\Rightarrow$ (1) $|f^{-1}(\tilde{y})|$ is the same for all $\tilde{y} \in U$.
(2) The set of regular values is open.

Lemma: Let M be a compact smooth manifold, N a smooth manifold of the same dimension and possibly with boundary. If $f, g: M \to N$ are smoothly homotopic and y is regular value of both f and g, then

$$|f^{-1}(y)| \bmod 2 = |g^{-1}(y)| \bmod 2.$$

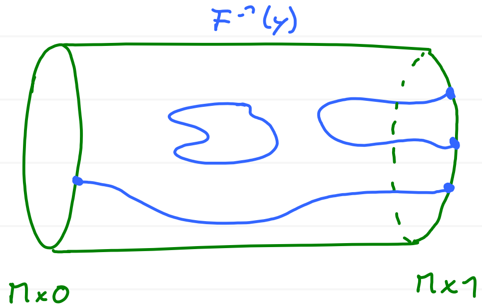
<u>proof</u>: Let $F: M \times [0,1] \to N$ be a smooth homotopy, s.t. $F_0 = f$, $F_1 = g$. Assume that y is a regular value for F. Then $F^{-1}(y)$ is a compact 1-dim. manifold with boundary

$$\partial(F^{-1}(y)) = F^{-1}(y) \cap \partial(M \times [0,1])$$
$$= F^{-1}(y) \cap (M \times \{0\} \cup M \times \{1\})$$
$$= f^{-1}(y) \times \{0\} \cup g^{-1}(y) \times \{1\}$$

Since $|\partial(F^{-1}(y))|$ must be even, the same has thus to be true for

$$|f^{-1}(y)| + |g^{-1}(y)|.$$

So $|f^{-1}(y)| \bmod 2 = |g^{-1}(y)| \bmod 2$



If y is not a regular value of F, take U_f and U_g open neighborhoods of y s.t. $|f^{-1}(y)|$ and $|g^{-1}(y)|$ are constant on U_f & U_g respectively. By Sard's thm. there is a $\tilde{y} \in U_f \cap U_g$ that is a regular value for F and for which we get

$$\text{even} = |\partial(F^{-1}(\tilde{y}))| = |f^{-1}(\tilde{y})| + |g^{-1}(\tilde{y})|$$
$$= |f^{-1}(y)| + |g^{-1}(y)| \qquad \square$$

<u>Def.</u>: Two diffeomorphisms $f, g: M \to N$ are called "smoothly isotopic" if there is a smooth homotopy $F: M \times [0,1] \to N$ s.t. $F_t: x \mapsto F(x,t)$ is a diffeomorphism for all $t \in [0,1]$.

Lemma: Let y, z be two points in a connected smooth manifold N. There is a diffeomorphism $h: N \to N$ s.t.

(i) $h(y) = z$

(ii) h is smoothly isotopic to the identity, i.e., there is an isotopy h_t s.t. $h_0 = h$ and $h_1 = id$.

(iii) $\{x \in N \mid h_t(x) \neq x\}$ is compact for all $t \in [0,1]$.

proof: The set of z's for which this is true for a given y forms an equivalence class. We will prove that this class is an open set. Since N is then a disjoint union of open sets, connectivity implies that there is only one class, which then consists of the entire manifold.

Using charts we can w.l.o.g. assume $N = \mathbb{R}^n$ and $y = 0$.
We use a basis in which z lies on the first coordinate axis, i.e.,
$z = (z_1, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$.

Choose $f_n \in C^\infty(\mathbb{R}^n, \mathbb{R})$ s.t. $f(0) = 1$ and $\|x\| \geq \varepsilon \Rightarrow f(x) = 0$.

Define $h_t : \mathbb{R} \times \mathbb{R}^{n-1} \ni (a,b) \mapsto \left(a + t f_{n-1}(b) f_1(a) z_1 ,\ b\right)$

Then (i) $h_1(0) = z$

(ii) $h_0(x) = x$

(iii) $h_t(x) = x$ if $\|x\| \geq \sqrt{2}\,\varepsilon$.

It remains to prove that h_t is a diffeomorphism on suitably chosen open sets. We show first that it is bijective. To this end consider

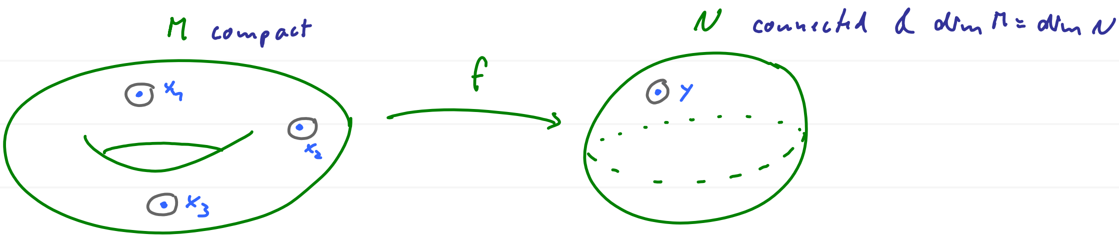
$$g: a \mapsto a + f_1(a)\, t\, f_{n-1}(b)\, z_1$$

Then $g'(a) = 1 + f_1'(a)\, t\, f_{n-1}(b)\, z_1 > 0$ for all $t \in [0,1]$ if $\|z\|$ is sufficiently small. Hence $g: \mathbb{R} \to \mathbb{R}$ is a diffeomorphism & therefore h_t bijective as long as z is close enough to 0.

For the derivative of h_t at $(a,b) \in \mathbb{R} \times \mathbb{R}^{n-1}$ we obtain:

$$d(h_t)_{(a,b)} = \left(\begin{array}{c|c} 1 + f_1'(a)\, t\, f_{n-1}(b)\, z_1 & * \; * \; * \\ \hline 0 & \mathbb{1} \end{array} \right)$$

So if $\|z\|$ is suf. small, then $d(h_t)_{(a,b)}$ is an isomorphism & the inverse function thm. implies that h_t^{-1} is smooth. $\square$



<u>Thm.:</u> Let M, N be smooth manifolds of the same dimension, M compact and N connected (and possibly with boundary). Then for a smooth map $f: M \to N$ with regular value y the "<u>mod 2 degree</u>" of f $\deg_2(f) := |f^{-1}(y)| \bmod 2$ is independent of the choice of the regular value y and depends only on the smooth homotopy class of f.

<u>proof:</u> Let y & z be two regular values and h an isotopy as in the previous Lemma so that $h_1(y) = z$.

Then z is a regular value of $h_1 \circ f$ and by using homotopy:

$$|f^{-1}(y)| \bmod 2 \underset{\uparrow}{=} |(h_1 \circ f)^{-1}(z)| \bmod 2 \underset{\uparrow}{=} |f^{-1}(z)| \bmod 2$$

$y = h_1^{-1}(z)$ $\qquad$ h_1 smoothly homotopic to $h_0 = id$

If f & g are smoothly homotopic, then $\deg_2(f) = \deg_2(g)$ if there is a common regular value, which always exists by Sard's thm.

$\square$