VIII. Sard's theorem

Reminder on sets of measure zero:

- $X \subseteq \mathbb{R}^n$ has measure zero if for any $\varepsilon > 0$ we can cover X by a set of cubes (or balls) so that their total volume is at most ε.
- Countable unions of sets of measure zero have measure zero.

(proof: Let $(X_i \subseteq \mathbb{R}^n)$ be such that $\mu(X_i) = 0$ $\forall i \in \mathbb{N}$ and pick $\varepsilon > 0$ & a sequence of cubes $Q_i^j \subseteq \mathbb{R}^n$ s.t. $\bigcup_j Q_i^j \supseteq X_i$ with $\sum_j \mu(Q_i^j) < 2^{-i}\varepsilon$.
Then $\bigcup_{i,j} Q_i^j \supseteq \bigcup_i X_i$ and $\sum_{ij} \mu(Q_i^j) < \varepsilon \sum_i 2^{-i} = \varepsilon$. $\square$)

- If $f \in C^1(U, \mathbb{R}^n)$ with $U \subseteq \mathbb{R}^n$ open, and $X \subseteq U$ has measure zero, then so does $f(X)$.

(proof idea: write $U \subseteq \bigcup_{i \in \mathbb{N}} B_i$ s.t. $\forall i \; \exists k_i \in \mathbb{R}$:
$$\| f(x) - f(y) \| \leq k_i \| x - y \| \quad \forall x, y \in B_i .$$
So if $Q \subseteq B_i$ is a cube of edge length λ, then $f(Q)$ is in a cube of edge length less than $k_i \sqrt{n}\, \lambda$. Thus $\mu(f(X \cap B_i)) = 0$ so that $\mu(f(X)) \leq \sum_i \mu(f(X \cap B_i)) = 0$.)

Def.: If $(M, \mathcal{A})$ is a smooth manifold, then $X \subseteq M$ is said to be a set of "measure zero" if $\forall (U, \varphi) \in \mathcal{A}$: $\varphi(X \cap U)$ has measure zero in $\mathbb{R}^{\dim M}$.

Remark: It is sufficient to check this for any atlas $\tilde{\mathcal{A}} \subseteq \mathcal{A}$, which can always be chosen such that it has only a countable number of charts.

Thm.: (Sard's theorem) If $f: M \to N$ is a smooth map between smooth manifolds, then the set of critical values of f in N has measure zero.

remark: note that the set of critical points in M, however, need not have measure zero. If for instance $f(x) = y$ is constant, then any $x \in M$ is a critical point.

proof: (for simplicity we assume $\dim M = \dim N = m$; see [Hirsch] for the general proof).

It suffices to consider a smooth map $f: [0,1]^m =: Q \to \mathbb{R}^m$.

Since $f \in C^1$ and Q is compact, we have Lipschitz continuity

$\forall x, x' \in Q: \; \| f(x) - f(x') \| \leq L \| x - x' \|$ for some $L \in [0, \infty)$.

If $f(c)$ is a critical value, then $df_c(Q)$ lives in a proper subspace of $\mathbb{R}^m$.

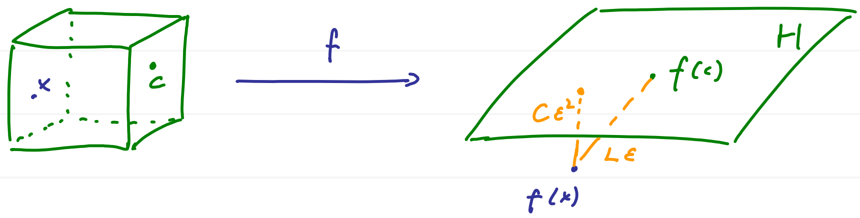
That is, there is a hyperplane $H \subseteq \mathbb{R}^m$ with $H \supseteq \{ y \in \mathbb{R}^m \mid y \in df_c(Q) + f(c) \}$.

By Taylor's thm. with remainder there is a $C \in [0, \infty)$ s.t. $\forall x \in Q$:

$$\inf_{y \in H} \| f(x) - y \| \leq \| f(x) - (f(c) + df_c(x - c)) \| \leq C \| x - c \|^2.$$

Thus, $\| x - c \| \leq \varepsilon \;\Rightarrow\; \| f(x) - y \| \leq C \varepsilon^2$ for some $y \in H$

$\wedge \; \| f(x) - f(c) \| \leq L \varepsilon$



The image of a cube that contains x & c and has edge length $\varepsilon \sqrt{m}$

is therefore contained in a cuboid of volume $(2C\varepsilon^2)(2L\varepsilon)^{m-1}$.

Now consider $Q = \bigcup_{i=1}^{\Delta^m} Q_i$ subdivided into Δ^m cubes of edge length Δ^{-1}.

Let γ be such that $i \in \gamma \Leftrightarrow Q_i$ contains a critical point. Then

$$\mu\Big[f\Big(\bigcup_{i \in \gamma} Q_i\Big)\Big] \underset{\varepsilon \leq \frac{\sqrt{m}}{\Delta}}{\leq} |\gamma| \left(2C \frac{m}{\Delta^2}\right)\left(2L \frac{\sqrt{m}}{\Delta}\right)^{m-1} \underset{|\gamma| \leq \Delta^m}{\leq} \frac{1}{\Delta}\, m^{\frac{m+1}{2}} C L^{m-1} 2^m \to 0$$

for $\Delta \to \infty$

$\square$

We can extend Sard's thm. to the case where M is allowed to be a smooth manifold with boundary:

Thm.: (Sard's thm. for manifolds with boundary)

Let $F: M \to N$ be a smooth map from a smooth manifold with boundary to a smooth manifold. Then the subset of N containing points that are either critical values of F or of $f := F|_{\partial M}$ has measure zero.

proof: If $x \in \partial M$ then $T_x(\partial M)$ is a subspace of $T_x M$ and df_x is the restriction of dF_x to that subspace. Hence,
rank $df_x = \dim N \Rightarrow$ rank $dF_x = \dim N$ and every critical value of F is either critical for $F^\circ := F|_{\mathrm{Int}(M)}$ or for f. So y is a critical value of F or f only if it is a critical value of $F^\circ := F|_{\mathrm{Int}\,M}$ or f.
The claim then follows from applying Sard's thm. to F° and f and using that the union of two sets of measure zero has again measure zero.

$\square$

Corollary: Let M be a smooth manifold with boundary, N a smooth manifold and $f: M \to N$ smooth. Then f has a regular value.

proof: If $\dim N > 0$, then this follows from Sard's theorem.
If $\dim N = 0$, i.e., N is discrete, then it is trivially true since rank $df_x = \dim N$ becomes void.

$\square$

IX. No-retraction & Brouwer's fixed point thm.

Lemma: Let M be a smooth manifold and $f: M \to \mathbb{R}$ a smooth map with regular value 0. Then $\{x \in M \mid f(x) \geq 0\}$ if non-empty is a smooth manifold with boundary equal to $f^{-1}(0)$ and has the same dim. as M.

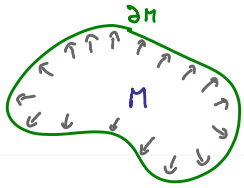
proof (sketch): By the constant rank thm., there are diffeomorphisms φ, ψ s.t. $\varphi \circ f \circ \psi^{-1}$ is locally a coordinate projection $(x_1, \ldots, x_m) \mapsto x_m$ for which the result is obvious. $\square$

Thm.: Let $f: M \to N$ be a smooth map between a smooth manifold with boundary M and a smooth manifold N with $m := \dim M > \dim N =: n$. If $y \in N$ is a regular value of both f and $f|_{\partial M}$, then $f^{-1}(\{y\})$ is a smooth manifold with boundary $\partial f^{-1}(\{y\}) = f^{-1}(\{y\}) \cap \partial M$ and $\dim f^{-1}(\{y\}) = m-n$.

proof: We know that $f^{-1}(\{y\}) \cap \mathrm{Int}(M)$ is a smooth manifold of dim. $m-n$ since $\mathrm{Int}(M)$ is a smooth manifold and y is a regular value of $f|_{\mathrm{Int}(M)}$.
So consider $z \in f^{-1}(\{y\}) \cap \partial M$ and assume w.l.o.g. $M = \mathbb{R}^m_+$.
There is an open $V \ni z$ in $\mathbb{R}^m$ and a smooth $\tilde{f}: V \to N$ s.t. $\tilde{f} = f$ on $V \cap \mathbb{R}^m_+$. By choosing V small enough we can guarantee that also $\tilde{f}$ has no critical points in V (by the lower semi-continuity of the rank), so y is a regular value of $\tilde{f}$ as well, and $\tilde{f}^{-1}(\{y\})$ is a smooth submanifold of V. Define $\pi: \tilde{f}^{-1}(\{y\}) \to \mathbb{R}$, $(x_1, \ldots, x_m) \mapsto x_m$.

$$\text{Then} \quad f^{-1}(\{y\}) \cap V = \tilde{f}^{-1}(\{y\}) \cap \mathbb{R}^m_+ = \{x \in \tilde{f}^{-1}(\{y\}) \mid \pi(x) \geq 0\}$$

is by the previous Lemma a smooth manifold with boundary $\pi^{-1}(0)$ leading to $\partial f^{-1}(\{y\}) = f^{-1}(\{y\}) \cap \partial M$. $\square$



Thm.: (no retraction theorem)

Let M be a compact smooth manifold with boundary $\partial M \neq \emptyset$. There is no smooth map $f: M \to \partial M$ that acts as the identity on ∂M.

proof: Suppose such a map existed.
By Sard's theorem f must have a regular value $y \in \partial M$.
Since y is a regular value for $f|_{\partial M} = id_{\partial M}$ as well, by the previous thm. $f^{-1}(\{y\})$ is a smooth 1-dim. manifold with boundary $\partial f^{-1}(\{y\}) = f^{-1}(\{y\}) \cap \partial M = \{y\}$ as $x \in \partial M \overset{f}{\mapsto} y \in \partial M$ implies $x = y$ due to $f|_{\partial M} = id$. Moreover, since $f^{-1}(\{y\})$ is a closed subset of a compact space it is itself compact.
Any compact smooth 1-manifold, however, has an even number of boundary points, leading to the sought contradiction. $\square$