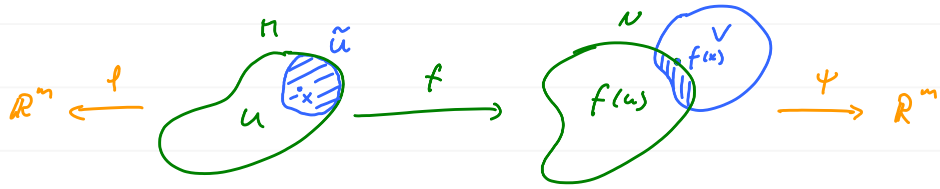
Lemma: (Smooth invariance of domain) Let M, N be smooth manifolds of equal dimension and $f: U \to f(U) \subseteq N$ a diffeomorphism from an open subset $U \subseteq M$. Then $f(U)$ is open in N.



proof: Since $f^{-1}: f(U) \to U$ is smooth, there is, for each $x \in U$, an open neighborhood V of $f(x)$ in N to which $f^{-1}|_{V \cap f(U)}$ can be extended to a smooth map $\hat{f}: V \to M$ s.t. $\hat{f}(y) = f^{-1}(y)$ $\forall y \in V \cap f(U)$.

Due to continuity of f, $f^{-1}(V) \cap U =: \tilde{u}$ is an open neighborhood of x and $\hat{f} \circ f|_{\tilde{u}} = f^{-1} \circ f|_{\tilde{u}} = id_{\tilde{u}}$.

Using charts to pull this identity back into Euclidean space we get:

$\varphi \hat{f} \psi^{-1} \circ \psi f \varphi^{-1} = id$ on the open set $\varphi(\tilde{u}) \subseteq \mathbb{R}^m$. Taking the derivative by using the chain rule this implies that $d(\psi f \varphi^{-1})_z$ is a vector space isomorphism for all $z \in \varphi(\tilde{u})$. Thus by the inverse function thm. there is an open neighborhood around any $z \in \varphi(\tilde{u})$ that is diffeomorphically mapped to an open image under $\psi f \varphi^{-1}$. Since ψ is continuous, $f(\tilde{u})$ and therefore also $f(U)$ is open in N. $\square$

remark:
- This is a smooth version of the (more complicated) "invariance of domain thm.", which states that if $U \subseteq \mathbb{R}^n$ is open an $f: U \to \mathbb{R}^n$ is a continuous injection, then $f(U)$ is open & $f: U \to f(U)$ is a homeomorphism. We'll come back to this later ...

The extension of the notion "diffeomorphism" to subsets of smooth manifolds, allows us, by applying it to $\mathbb{R}^n_+ \subseteq \mathbb{R}^n$, to extend the notion "smooth manifold" in a straight forward manner to "smooth manifold with boundary".
If $(M, \mathcal{A})$ is an n-dim. smooth manifold with boundary, then

$$\partial M := \{ x \in M \mid \exists (U, \varphi) \in \mathcal{A} : x \in U \wedge \varphi(x)_n = 0 \}$$

$$\mathrm{Int}(M) := \{ x \in M \mid \exists (U, \varphi) \in \mathcal{A} : x \in U \wedge \varphi(x)_n > 0 \}$$

Again $\mathrm{Int}(M)$ and ∂M are smooth manifolds of dim. $\dim(M)$ and $\dim(M)-1$ resp.
A priori, it is not obvious that $\partial M \cap \mathrm{Int}(M) = \emptyset$. This is, however, a consequence of the smooth invariance of domain. If we set $M = N = \mathbb{R}^n$ we obtain:

<u>Corollary:</u> If $U \subseteq \mathbb{R}^n_+ \setminus \partial \mathbb{R}^n_+$ is open and diffeomorphic to $V \subseteq \mathbb{R}^n_+$, then $V \cap \partial \mathbb{R}^n_+ = \emptyset$.

<u>Corollary:</u> If M is a smooth manifold with boundary, then $M \setminus \partial M = \mathrm{Int}(M)$

<u>proof:</u> Assume this is not the case, i.e., for some $x \in M$ we find charts $(\varphi_i, U_i) \in \mathcal{A}$ s.t. $\varphi_1(x) \in \partial \mathbb{R}^n_+$ and $\varphi_2(x) \notin \partial \mathbb{R}^n_+$ with $U_1 \cap U_2 \ni x$.
Due to continuity of φ_2 there is an open neighborhood $\tilde{U}_2$ of x s.t. $\varphi_2(\tilde{U}_2) \cap \partial \mathbb{R}^n_+ = \emptyset$. Define $W := U_1 \cap U_2 \cap \tilde{U}_2 \ni x$.
Since M is a smooth manifold with boundary, we have

$$f := \varphi_1 \circ \varphi_2^{-1} : \varphi_2(W) \subseteq \mathbb{R}^n_+ \to \varphi_1(W) \subseteq \mathbb{R}^+_n \text{ is a diffeomorphism}$$

However, $f(\varphi_2(W)) \cap \partial \mathbb{R}^n_+ \neq \emptyset$ together with $\varphi_2(W) \subseteq \mathbb{R}^n_+ \setminus \partial \mathbb{R}^n_+$ contradicts the previous corollary. $\square$

<u>Corollary:</u> If $f : M \to N$ is a diffeomorphism between two smooth manifolds with boundary, then

$$f(\mathrm{Int}(M)) = \mathrm{Int}(N)$$
$$\text{and} \quad f(\partial M) = \partial N .$$

<u>proof:</u> assume that $f(\mathrm{Int}(M)) \cap \partial N \neq \emptyset$. Then there would be an open $U \subseteq \mathbb{R}^n_+ \setminus \partial \mathbb{R}^n_+$, which would be diffeomorphically mapped onto $V := \psi \circ f \circ \varphi^{-1}(U)$ s.t. $V \cap \partial \mathbb{R}^n_+ \neq \emptyset$. $\square$

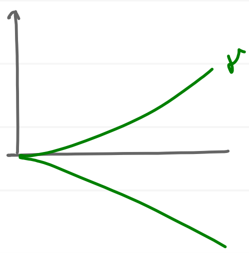
Def.: Let $f: M \to N$ be a smooth map between smooth manifolds $(M, \mathcal{A})$ and $(N, \mathcal{B})$.

- The "rank" of f at $x \in M$ is the rank of $d(\psi f \varphi^{-1})_{\varphi(x)}$, where $(U, \varphi) \in \mathcal{A}$, $(V, \psi) \in \mathcal{B}$ with $x \in U$, $f(x) \in V$.
- f is an "immersion" if $\operatorname{rank} f = \dim(M)$ everywhere.
- f is an "embedding" if it is an immersion and $f: M \to f(M)$ is a homeomorphism.
- f is a "submersion" if $\operatorname{rank} f = \dim(N)$ everywhere.
- $y \in N$ is called a "regular value" of f if $\operatorname{rank} f = \dim(N)$ $\forall x \in f^{-1}(y)$. Otherwise y is called "critical value" of f.

remarks:
- $y \notin f(M) \Rightarrow y$ is a regular value of f.
- immersions / submersions are locally injective / surjective.

examples:
- curves $\gamma: (-1,1) \subseteq \mathbb{R} \to \mathbb{R}^n$ are immersions iff $\forall x \in (-1,1): \gamma'(x) \neq 0$

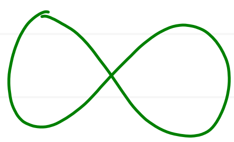
  i.e. $\gamma(t) := \begin{pmatrix} t^2 \\ t^3 \end{pmatrix}$ is not an immersion ("sharp edge" at 0)



- a Lemniscate (figure eight) $\gamma: S^1 \to \mathbb{R}^2$,

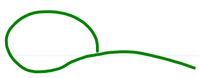
  $\gamma(\cos t, \sin t) = (\sin t, \sin 2t)$

  is a non-injective immersion



- an injective immersion, which is not an embedding: e.g. an injective curve with contact point,

  $\gamma: (-1,1) \to \mathbb{R}^2$ with $\lim_{t \to 1} \gamma(t) = \gamma(0)$.



  For $U := (-\varepsilon, \varepsilon)$, the image $\gamma(U)$ is not open & thus γ not a homeomorphism.

- the inclusion map $\iota: S^1 \to \mathbb{R}^2$ is an embedding,
- the map $f: \mathbb{R}^2 \setminus \{0\} \to S^1$, $f(x) := \frac{x}{\|x\|}$ is a submersion.