I. Reminder of topological spaces

I.1 Basic definitions

Def.:

- A topology T on a set X is a family of subsets of X s.th.
  - (i) $\emptyset, X \in T$
  - (ii) $U, V \in T \Rightarrow U \cap V \in T$
  - (iii) $(U_\lambda)_{\lambda \in \Lambda}$ with $U_\lambda \in T \Rightarrow \bigcup_{\lambda \in \Lambda} U_\lambda \in T$
- (X, T) is then called a topological space and $U \in T$ are called open sets.
- $C \subseteq X$ is called closed if $X \setminus C$ is open.
- The closure $\bar{A}$ of $A \subseteq X$ is the smallest closed subset of X s.th. $A \subseteq \bar{A}$ (i.e., the intersection of all closed sets containing A)
- The interior A° of $A \subseteq X$ is the largest open subset of X s.th. $A^\circ \subseteq A$. (i.e., the union of all open sets contained in A)
- $A \subseteq X$ is said to be dense in X if $\bar{A} = X$.
- $U \subseteq X$ is called a neighborhood of $x \in X$ if $\exists V \in T : x \in V \subseteq U$.
- A family $\mathcal{B} \subseteq T$ is called a basis of the topology T if $\forall U \in T \; \exists \Lambda \subseteq \mathcal{B} : U = \bigcup_{V \in \Lambda} V$ (i.e. open sets are unions of sets from the basis)

Def.: (Properties of topological spaces)

- (X, T) is called connected if $X = X_1 \cup X_2$, $\emptyset \neq X_i \in T$ implies $X_1 \cap X_2 \neq \emptyset$.
- (X, T) is called a Hausdorff space if $\forall x, y \in X$ with $x \neq y$ there are neighborhoods U_x, U_y of x and y respectively, s.th. $U_x \cap U_y = \emptyset$ (i.e. distinct points in X can be seperated by open sets)

- (X,T) is called <u>second countable</u> if there is a countable basis of T.

<u>Examples</u>:

- <u>Metric topology</u> for a metric space (X,d):
  Let $B_r(x) := \{y \in X \mid d(x,y) < r\}$ and define
  $$T := \{V \subseteq X \mid \forall x \in V\ \exists r > 0 : B_r(x) \subseteq V\}$$
  - $\{B_r(x)\}_{r \in \mathbb{R}, x \in X}$ is a basis for the metric topology.
  - (X,T) is a Hausdorff space
  - (X,T) is second countable iff X has a countabable dense subset. $\rightsquigarrow$ Exercises
- <u>Trivial topology</u> of X:
  $$T := \{\emptyset, X\}$$
  - (X,T) is not a Hausdorff space if $|X| \geq 2$.

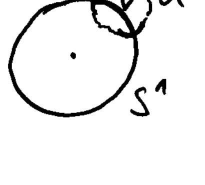
$\rightsquigarrow$ More examples in the exercises.

I.2. Constructing new topological spaces from old ones

<u>Def.</u>:

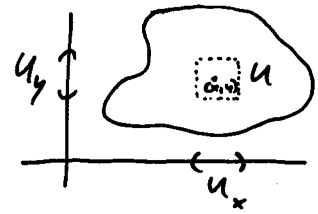
- The <u>relative topology</u> (or <u>subspace topology</u>) of a subset $A \subseteq X$ of a top. space (X,T) is defined by
  $$R := \{V \subseteq A \mid \exists U \in T : U \cap A = V\}$$
  The open sets of the relative topology are also called <u>relatively open</u>.

  <u>Example</u>: The sphere $S^n := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ inherits a topology from Euclidean space $\mathbb{R}^{n+1}$.



- The <u>product topology</u> of (X,T) and (Y,R) is defined as

$$\{ U \subseteq X \times Y \mid \forall (x,y) \in U \text{ there are open neighborhoods } U_x \in T,\ U_y \in R : U_x \times U_y \subseteq U \}$$



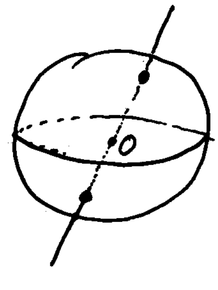
- The <u>quotient topology</u> of a quotient $X/\sim$ of (X,T) is defined as

$$Q := \{ V \subseteq X/\sim \mid q^{-1}(V) \in T \} \text{ if } q : X \to X/\sim$$

s.th. $x \sim x' \Leftrightarrow q(x) = q(x')$.

<u>Examples:</u>

- <u>Projective spaces</u> for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$:

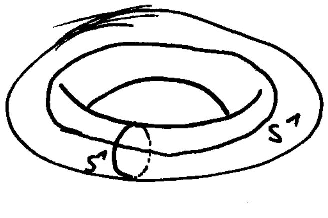


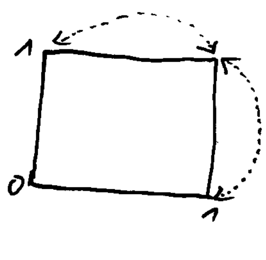
$\mathbb{K}P^n := (\mathbb{K}^{n+1} \setminus \{0\})/\sim$ where

$$x \sim y \Leftrightarrow \exists \lambda \in \mathbb{K} \setminus \{0\} : x = \lambda y$$

Alternatively, for instance $\mathbb{R}P^n := \{ \{x, -x\} \mid x \in S^n \}$ and we can set $q : S^n \to \mathbb{R}P^n,\ x \mapsto [x]$ s.th. $q^{-1}([x]) = \{-x, x\}$.

- <u>Torus</u> T^2 can be regarded as product space $S^1 \times S^1$, as a quotient of $[0,1] \times [0,1]$ by identifying parallel edges or as a subspace $T^2 \subseteq \mathbb{R}^3$.
Fortunately product, quotient & subspace topologies coincide here.





I.3. Compactness, convergence & continuity

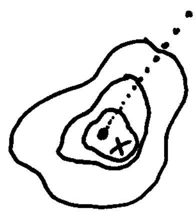
Def.: Let (X,T) be a topological space.

- A subset $Y \subseteq X$ is called <u>compact</u> if any open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ with $U_\lambda \in T$ and $\bigcup_{\lambda\in\Lambda} U_\lambda \supseteq Y$ has a finite subcover $\bigcup_{i=1}^{N} U_{\lambda_i} \supseteq Y$.

  <u>Note</u>: Closed subsets of compact sets are compact.

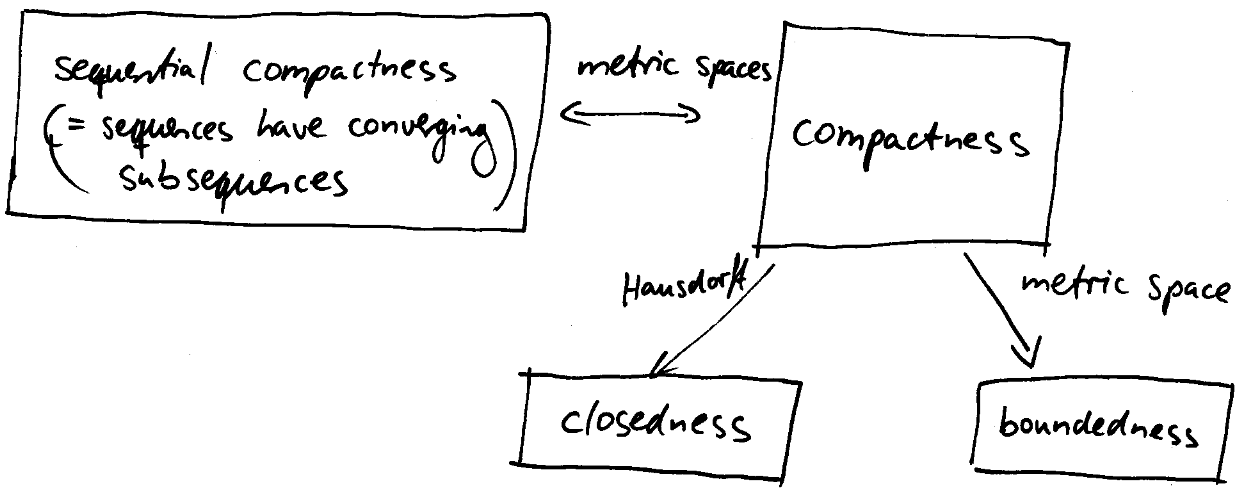
- A sequence $(x_n)_{n\in\mathbb{N}}$ is said to <u>converge</u> to $x \in X$ if

  $$\forall U \in T : \left(x \in U \Rightarrow \exists m \in \mathbb{N} : \forall n \geq m : x_n \in U\right)$$



<u>Note</u>:
- In a metric space the closure $\bar{A}$ of A is the set of limits of all sequences in A.
- In Hausdorff spaces limits are unique.

<u>Remark</u>:



(X,T) (Y,R)

<u>Def.:</u>

- A map $f: X \to Y$ between topological spaces is said to be <u>continuous</u> if $f^{-1}(U) \in T \ \forall U \in R$.
  (i.e. preimages of open sets are open)

  <u>Note:</u>
  - <u>Equivalently:</u> preimages of closed sets are closed
  - If f is continuous then $x_n \to x$ implies $f(x_n) \to f(x)$.
    In metric spaces this is equivalent to continuity.
  - $C(X,Y) :=$ set of all continuous functions from X to Y.

- A map $f: X \to Y$ between topological spaces is called <u>open</u> (<u>closed</u>) if it maps open sets to open sets (resp. closed sets to closed sets).

- A map $f: X \to Y$ between topological spaces is called a <u>homeomorphism</u> if
  (i) f is bijective
  (ii) f is continuous
  (iii) f^{-1} is continuous

  Equivalently if f is a continuous <u>and</u> open bijection.

  <u>Note:</u> $f: [0,2\pi) \to S^1$, $t \mapsto (\cos(t), \sin(t))$
  is continuous and bijective, but <u>not</u> a homeomorphism.

- Topological spaces are called <u>homeomorphic</u> if there is a homeomorphism between them.

Remark:

- If $f \in C(X, Z)$ and $Y \subseteq X$ is compact, then $f(Y)$ is compact
  - $\Rightarrow$ $f \in C(X, \mathbb{R})$ has min & max on compact subsets.
  - $\Rightarrow$ $[0, 2\pi)$ and S^1 cannot be homeomorphic.
  - $\Rightarrow$ $\mathbb{K}P^n$ is compact.
- If $f \in C(X, Z)$ is a bijection from a compact to a Hausdorff space, then f is a homeomorphism.

II. Topological manifolds

Def.: A second countable Hausdorff space (M,T) is called a <u>topological manifold</u> of dimension $m \in \mathbb{N}_0$ if it is locally homeomorphic to $\mathbb{R}^m$.

- "<u>locally homeomorphic to $\mathbb{R}^m$</u>" means that for every $x \in M$ there is an open neighborhood $U \in T$ and a homeomorphism $\varphi : U \subseteq M \to V \subseteq \mathbb{R}^m$.
  - The pair (U, φ) is called a <u>chart</u>
  - A family $\{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$ of charts is called an <u>atlas</u> for M if $\bigcup_{\lambda \in \Lambda} U_\lambda \supseteq M$.
  - $\varphi_1, \ldots, \varphi_m$ are called <u>coordinates</u> and φ^{-1} a <u>parametrization</u>

<u>Remark:</u>

- "<u>Continuous topology</u>" studies properties of topological manifolds invariant under homeomorphisms.
- "<u>Differential topology</u>" studies properties of smooth manifolds invariant under diffeomorphisms.
  - $\hookrightarrow$ We will learn later what that means.