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Zentrum Mathematik

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Differential Topology

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Tutoraufgaben

1. Classification of smooth, compact 1-Manifolds

Let $(M, \mathcal{A})$ be a compact, smooth and connected 1-dimensional manifold with boundary. Show that $(M, \mathcal{A})$ is either diffeomorphic to a circle or to a closed interval $[a, b] \subset \mathbb{R}$.

Note: This shows that every compact smooth 1-manifold is build up from connected components each diffeomorphic to a circle or a closed interval.

To prove this result, show the following steps:

(a) Consider a smooth function $f : M \to \mathbb{R}$ which critical points are non-degenerate (such a function is called a **Morse function**). Let $\mathcal{B}$ denote the set of boundary points of M, and $\mathcal{C}$ the set of critical points of f.
Show that $M \setminus (\mathcal{B} \cup \mathcal{C})$ consists of a finite number of connected 1-manifolds, which we will call $L_1, \ldots, L_N \subset M$.

(b) Prove that $f : L_k \to f(L_k)$ is a diffeomorphism from L_k to an open interval $f(L_k) \subset \mathbb{R}$.

(c) Let $L \subset M$ be a subset of a smooth 1-dimensional manifold M. Show that if L is diffeomorphic to an open interval in $\mathbb{R}$, then there are at most two points $a, b \in \overline{L}$ such that $a, b \notin L$.

(d) Consider a sequence $(L_1, \ldots, L_k) \in \{L_1, \ldots, L_N\}^k$. Such a sequence is called a **chain** if for every $j \in \{1, \ldots, k-1\}$ the closures $\overline{L_j}$ and $\overline{L_{j+1}}$ have a common boundary point.
Show that a chain of maximal length k contains all of the 1-manifolds $L_1, \ldots, L_N$ defined above.

(e) To finish the prove, we will need a technical lemma. Let $g : [a, b] \to \mathbb{R}$ be a smooth function with positive derivative on $[a, b] \setminus \{c\}$ for $c \in (a, b)$. Then there exists a smooth function $\tilde{g} : [a, b] \to \mathbb{R}$ with positive derivative on the whole of $[a, b]$ such that $g = \tilde{g}$ on a neighborhood of a and on a neighborhood of b.

(f) Finally use the lemma from the previous exercise to show that if $\overline{L_1}$ and $\overline{L_N}$, contained in the maximal chain, have a common boundary point $(M, \mathcal{A})$ is diffeomorphic to a circle. Otherwise $(M, \mathcal{A})$ is diffeomorphic to a closed interval.

Hausaufgaben

3.1. Real and Complex Projective spaces are smooth manifolds

In Homework exercise 1.3 we proved, that $\mathbb{R}P^n$ is a topological manifold.

a) Show that $\mathbb{C}P^n$ is a topological manifold.
Hint: Go through the proof for $\mathbb{R}P^n$ and make minor changes. Identify $\mathbb{C} \simeq \mathbb{R}^2$ and use charts of the form

$$\phi_i(x) = \left(\mathrm{Re}\left(\frac{\tilde{x}_1}{\tilde{x}_i}\right), \mathrm{Im}\left(\frac{\tilde{x}_1}{\tilde{x}_i}\right), \ldots, \mathrm{Re}\left(\frac{\tilde{x}_{i-1}}{\tilde{x}_i}\right), \mathrm{Im}\left(\frac{\tilde{x}_{i-1}}{\tilde{x}_i}\right), \mathrm{Re}\left(\frac{\tilde{x}_{i+1}}{\tilde{x}_i}\right), \mathrm{Im}\left(\frac{\tilde{x}_{i+1}}{\tilde{x}_i}\right) \ldots\right)$$

for a representative $\tilde{x}$ of x.

b) Show that $\mathbb{R}P^n$ and $\mathbb{C}P^n$ are smooth manifolds.

3.2. Complex Projective Space

Prove that $\mathbb{C}P^1$ and S^2, with the smooth structures given in lecture/exercise, are diffeomorphic.

Hint: Show that the map $f : S^2 \to \mathbb{C}P^1$ defined as

$$f(x_0, x_1, x_2) := \begin{cases} \left[\frac{x_1 - ix_2}{1-x_0}, 1\right], & x_0 \neq 1 \\ \\ \left[1, \frac{x_1+ix_2}{1+x_0}\right], & x_0 \neq -1 \end{cases}$$

from the lecture is a diffeomorphism between S^2 and $\mathbb{C}P^1$.

3.3. Manifolds with Boundary

Let M denote an $n-$dimensional topological manifold with boundary such that $\partial M \neq \emptyset$.

(a) Show that the manifold interior $\operatorname{Int}(M)$ is an $n-$dimensional topological manifold without boundary.

(b) Show that the manifold boundary ∂M is an $(n-1)-$dimensional topological manifold without boundary.

3.4. Immersions and Embeddings

Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ denote smooth manifolds of dimensions $\dim(M) = m$ and $\dim(N) = n \geq m$.

(a) Let $f : M \to N$ be an immersion between the manifolds. Show that for all $x \in M$ there is a neighborhood $x \ni U \subset M$ such that $f|_U : U \to f(U)$ is an embedding, i.e. an immersion mapping U homeomorphically to $f(U)$.
("**An immersion is locally an embedding**".)

(b) Let $(M, \mathcal{A}), (N, \mathcal{B})$ denote smooth manifolds. Show that $f : M \to N$ is an embedding iff $f : M \to f(M)$ is a diffeomorphism.

Hint: Use the constant rank theorem.

3.5. System of Equalities

Is the set $S := \{x \in \mathbb{R}^3 | \sum_{i=1}^3 x_i^3 = 1 \wedge \sum_{i=1}^3 x_i = 0\}$ a smooth submanifold of $\mathbb{R}^3$?