sheet 3

exercise 1

1) Let's first examine $S_{2\times2}$: every hermitian matrix can be always written using the Pauli basis $(\mathbb{1}, \sigma_x, \sigma_y, \sigma_z)$, where

$$\mathbb{1} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad \sigma_x = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \quad \sigma_y = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} \quad \sigma_z = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

so, in general

$$\rho = \alpha \mathbb{1} + \sum_{\gamma = x,y,z} x_\gamma \sigma_\gamma = \begin{vmatrix} \alpha + x_z & x_x - i x_y \\ x_x + i x_y & \alpha - x_z \end{vmatrix}$$

we require $tr(\rho) = 2\alpha = 1$, hence $\alpha = \frac{1}{2}$.

For convenience, define also the x_γ's divided by 2, so that

$$\rho = \frac{1}{2}\left(\mathbb{1} + \sum_{\gamma=x,y,z} x_\gamma \sigma_\gamma\right) = \frac{1}{2}\left(\mathbb{1} + \vec{x}\cdot\vec{\sigma}\right)$$

where $\vec{x} = (x_x, x_y, x_z)$ can be seen as a vector in $\mathbb{R}^3$.

$\rho \geq 0 \iff eig(\rho) \geq 0$

$\det(\rho - \mu\mathbb{1}) = 0$ yields $\mu = \frac{1}{2}\left(1 \pm \sqrt{\sum_\gamma x_\gamma^2}\right) \geq 0$

$$\Updownarrow$$

$$\|\vec{x}\| \leq 1$$

hence, this suggests the definition.

$$\phi : S_{BLOCH} \longrightarrow S_{2\times2}$$
$$\vec{x} \longmapsto \frac{1}{2}\left(\mathbb{1} + \vec{x}\cdot\vec{\sigma}\right)$$

obviously, $\phi(\vec{x}) \in S_{2\times2} \quad \forall \vec{x} \in S_{BLOCH}$

and as proven before, $\phi^{-1}(\rho(\vec{x})) \in S_{BLOCH}$

therefore ϕ is an isomorphism. $\square$

2) $\rho \geq 0$, hence $\lambda_i \geq 0$ $\forall$ eigenvalues of ρ

~~[illegible]~~

moreover, $tr(\rho) = \sum_i \lambda_i = 1$

hence, the only elements not representable as a convex combinations of other elements are those with $\lambda_j = 1$, $\lambda_{i \neq j} = 0$ for $j = 1, \dots, dim(\rho)$.

They correspond to the PURE STATES.

$\square$

3) $0 \leq A \leq 1\!\!1$

from $A \geq 0$ we have $eig(A)_i = \lambda_i \geq 0$ $\forall i = 1, \dots, dim(A)$

$A \leq 1\!\!1$; $1\!\!1 - A \geq 0$

for every λ_i, suppose the eigenvalue equation is

$A x_i = \lambda_i x_i$

then, $(1\!\!1 - A) x_i = (1 - \lambda_i) x_i \implies \lambda_i \leq 1$ $\forall i$

the whole condition is then

$0 \leq \lambda_i \leq 1$ $\forall i$

The extremal points are then all the matrices whose eigenvalues are only 0 and 1.

$\square$

exercise ~~3~~2

$\mathcal{C}$ is closed and bounded therefore, as we are in finite dimensions, it is compact.

This means that $\forall a \in \mathbb{R}^n$, $a \notin \mathcal{C}$ we can apply corollary 1.19 in the lecture notes:

$\forall a \in \mathbb{R}^n$, $a \notin \mathcal{C}$, $\exists$ hyperplane separating $\mathcal{C}$ and a strongly.

This hyperplane separates $\mathbb{R}^n$ into two half-spaces, that can be characterised by two inequalities. Call H_a the half-space s.t. $\mathcal{C} \subset H_a$.

Now define the set:

$$D := \bigcap_{a \notin \mathcal{C}} H_a \quad , \text{ where the intersection is } \forall a \notin \mathcal{C}$$

Then $\mathcal{C} \subseteq D$ by definition and, as $\mathcal{C}$ is closed, $\mathcal{C} = D$.

$\square$

exercise 3

1) Take $S \subset \mathbb{R}^n$ convex. With this assumption, we want to prove:

$$f \text{ convex} \iff f((1-\lambda)x+\lambda y) \leq (1-\lambda)f(x)+\lambda f(y)$$
$$\forall x,y \in S;\ \forall \lambda \in [0,1]$$

$\Rightarrow$) f convex $\iff$ ep(f) convex

let $(x, f(x)), (y, f(y)) \in$ ep(f)

by convexity of ep(f):

$$\big((1-\lambda)x+\lambda y,\ \underbrace{(1-\lambda)f(x)+\lambda f(y)}_{\geq f((1-\lambda)x+\lambda y)}\big) \in \text{ep}(f) \quad \forall \lambda \in [0,1]\ \forall x,y \in S$$

$\Leftarrow$) let the inequality hold.

take $(x,\mu_x), (y,\mu_y) \in$ ep(f). S convex, so

$$\big((1-\lambda)x+\lambda y,\ \mu \geq f((1-\lambda)x+\lambda y)\big) \in \text{ep}(f) \quad \forall x,y,\lambda$$

as the inequality for f holds, we can choose

$$\mu = (1-\lambda)\mu_x+\lambda\mu_y \geq (1-\lambda)f(x)+\lambda f(x) \geq f((1-\lambda)x+\lambda y)$$

hence ep(f) is convex

$\Downarrow$

f convex

$\square$

2) Let's take a 1-dimensional open subset $(a,b) \subset$ ~~S~~ S, then the result can be extended to higher-dimensional sets.
Take $f : (a,b) \to \mathbb{R}$ convex ; take also $x, z \in (a,b)$
$x < z$; then $\exists\, y \in (a,b) : x < y < z$.

$$f(y) = f((1-\lambda)x + \lambda z) \underset{\substack{\uparrow \\ \text{ex. } 3.1}}{\leq} (1-\lambda) f(x) + \lambda f(z)$$

$$\Longleftrightarrow \quad f(y) - f(x) \leq -\lambda \left(f(x) - f(z) \right) \qquad , \ \lambda \in [0,1)$$

let's take $\lambda := -\dfrac{y-x}{x-z}$; then

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(x) - f(z)}{x - z} =: c \quad \text{constant} \ \forall y \in (x,z)$$

hence, we can always find a $c < \infty$ s.t.

$$| f(y) - f(x) | < c \, |x - y|$$

$\Longrightarrow$ f is Lipschitz continuous
$\Downarrow$
f continuous

$\square$

exercise 4

1) $V \subseteq \mathbb{R}^n$, $\mathcal{C} \subset V$ convex, absorbing, balanced

First, note that ρ is well-defined as $\mathcal{C}$ is absorbing.

To see that ρ is a seminorm, we have to prove:

- $\rho(\lambda x) = |\lambda| \rho(x) \quad \forall x \in V, \forall \lambda \in \mathbb{R}$
- $\rho(x+y) \leq \rho(x) + \rho(y) \quad \forall x, y \in V$

<u>homogeneity</u>: take $\lambda \geq 0$

$$\rho(\lambda x) = \inf\{t \in \mathbb{R}^+ \mid \underbrace{\frac{\lambda x}{t}}_{\text{call } \frac{\lambda}{t} = \frac{1}{t'}} \in \mathcal{C}\} = \inf\{t' \mid \frac{x}{t'} \in \mathcal{C}\} = \rho(x)$$

$$\Rightarrow \rho(\lambda x) = \lambda \rho(x)$$

now take $\lambda < 0$, and note that $\mathcal{C}$ balanced means $\rho(-x) = \rho(x)$. Then

$$\rho(-|\lambda| x) = \rho(|\lambda| x) = |\lambda| \rho(x)$$

<u>triangle inequality</u>: let's exploit the convexity in $\mathcal{C}$:

take $x, y \in \mathcal{C}$ and define $\rho(x) =: t$, $\rho(y) =: s$

$$\Rightarrow (1-\lambda)\frac{x}{t} + \lambda \frac{y}{s} \in \mathcal{C}$$

choose $\lambda = \frac{s}{s+t}$, so that: $\frac{x+y}{s+t} \in \mathcal{C}$

now consider:

$$\rho(x+y) = \inf\{\tilde{\lambda} \mid \frac{x+y}{\tilde{\lambda}} \in \mathcal{C}\} \leq s+t = \rho(x) + \rho(y)$$

$\square$

2) define the p-norm on $V = \mathbb{R}^n$: $\| x \|_p := \left(\sum_i |x_i|^p \right)^{1/p}$

define the unit ball $B_1^p(0) := \{ x \in V \mid \|x\|_p \leq 1 \}$

convexity of B : B convex if it contains all the possible convex combinations of its points; take

$z = \sum_j \lambda_j x_j \quad , \quad x_j \in B \ \forall j \ , \ \sum_j \lambda_j = 1 \ , \ \lambda_j \in [0,1]$

$\| z \|_p \leq \sum_j \lambda_j \underbrace{\| x_j \|_p}_{\leq 1} \leq \sum_j \lambda_j = 1$

$\Rightarrow z \in B \quad \forall$ convex comb. $\Rightarrow B$ convex

B absorbing : $x \in V$, $t \in \mathbb{R}^+$

$tx \in B \iff \| tx \|_p = t \| x \|_p \leq 1$

true for $t \leq \frac{1}{\| x \|_p}$, $\forall x \in V$

B balanced : recall that B is convex

take $x \in B \quad \Rightarrow \quad \| x \|_p = \| -x \|_p$

$x \in B \iff -x \in B$

moreover, $\rho(x) = \| x \|_p$ as :

$\rho(x) = \inf \{ t \in \mathbb{R}^+ \mid x \in tB \} =$

$= \inf \{ t \in \mathbb{R}^+ \mid \| x \|_p \leq t \} = \| x \|_p$

$\square$