# sheet 12

## exercise 1

1) Let $X = (x_1, x_2, \dots, x_n)$ be a set, to which we associate probabilities $p(x_j)$, $j = 1, \dots, n$.

$$\Rightarrow H(X) = -\sum_{x \in X} p(x) \log_2 p(x) \geq 0$$

since $p(x) \in [0,1] \Rightarrow \log_2 p(x) \leq 0 \quad \forall x \in X$ $\square$

2) Let $H(X,Y)$ be the entropy of the joint prob. density of $(X,Y)$. Recall that $p(x) = \sum_{y \in Y} p(x,y)$ (and similarly for $p(y)$); then

$$H(X,Y) - H(X) - H(Y) =$$

$$= -\sum_{\substack{x \in X \\ y \in Y}} p(x,y) \left[ \log_2(p(x,y)) - \log_2(p(x)) - \log_2(p(y)) \right] =$$

$$= \sum_{x,y} p(x,y) \log_2 \left[ \frac{p(x)\, p(y)}{p(x,y)} \right] \leq$$

(use Jensen's ineq. with $\log_2$, that is concave)

$$\leq \log_2 \Bigg[ \underbrace{\sum_{x,y} \cancel{p(x,y)} \frac{p(x)\, p(y)}{\cancel{p(x,y)}}}_{1} \Bigg] = 0$$

$\square$

3) we can use the same approach as before:

$$H(X,Y,Z) + H(Y) - H(X,Y) - H(Y,Z) =$$

$$= \sum_{\substack{x \in X \\ y \in Y \\ z \in Z}} p(x,y,z) \cdot \log_2 \left[ \frac{p(x,y)\ p(y,z)}{p(x,y,z)\ p(y)} \right] \leq (\text{Jensen})$$

$$\leq \log_2 \left[ \sum_{y} \frac{\overbrace{\sum_x p(x,y)}^{p(y)} \cdot \overbrace{\sum_z p(y,z)}^{p(y)}}{p(y)} \right] = 0$$

$\square$

4) Start from the right side of the eq.:

$$\sum_{i=1}^{n} H(X_i \mid Y, X_1, \ldots, X_{i-1}) =$$

$$= \sum_{i=1}^{n} \left[ H(X_1, X_2, \ldots, X_i, Y) - H(X_1, \ldots, X_{i-1}, Y) \right] =$$

= all these terms cancel each other, apart from

$$= -H(Y) + H(X_1, \ldots, X_n, Y) =$$

$$= H(X_1, \ldots, X_n \mid Y)$$

$\square$

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## exercise 2

1) Let $\rho^A$ be a density operator in a Hilbert space $H_A$. Then

$$S(A) := -\mathrm{tr}\{\rho^A \ln \rho^A\} =$$

(diagonalize $\rho^A$ s.t. $\rho^A = U\Delta U^\dagger$)

$$\cancel{= -\mathrm{tr}\{U\rho^A(\ln\rho^A)U^\dagger\} = -\mathrm{tr}\{\rho^A \ln \rho^A\}}$$

$$= -\mathrm{tr}\{U\Delta \ln \Delta\, U^\dagger\} = \quad \text{CICLICITY}$$

$$= -\mathrm{tr}\{\Delta \ln \Delta\} \geq 0$$

since the eigenvalues $\lambda_i(\rho^A) \in [0,1]$, $\forall i$ $\square$

2) Let's exploit the inequality:

$$\mathrm{tr}\{\rho \ln \rho - \rho \ln \sigma\} \geq 0 \qquad \forall \rho, \sigma \text{ states}$$

Consider $\rho^{AB} \in B(H_A \otimes H_B)$, and let $\rho^A, \rho^B$ be the reduced density matrices of systems A and B resp.

Choose $\rho := \rho^{AB}$ and $\sigma := \rho^A \otimes \rho^B$, then

$$-S(A,B) - \underbrace{\mathrm{tr}\{\rho^{AB} \ln(\rho^A \otimes \rho^B)\}} \geq 0$$

$$= \mathrm{tr}\{\rho^{AB}(\ln(\rho^A \otimes 1\!\!1) + \ln(1\!\!1 \otimes \rho^B))\}$$

now, note that $\forall \Delta, \Delta'$ diagonal matrices we have

$$\ln(\Delta \otimes \Delta') = (\ln \Delta) \otimes 1\!\!1 + 1\!\!1 \otimes (\ln \Delta')$$

Therefore, through diagonalising $\rho^A$ ~~[illegible]~~, one gets:

$$\mathrm{tr}_{AB}\{\rho^{AB} \cdot (\ln(\rho^A \otimes \mathbb{1}))\} = \mathrm{tr}_{AB}\{\rho^{AB} \cdot (\ln \rho^A) \otimes \mathbb{1}\} =$$

$$= \mathrm{tr}_A\{\rho^A \ln \rho^A\} = -S(A)$$

$$\Rightarrow \quad S(A,B) \leq S(A) + S(B) \qquad \square$$

3) Let $\rho^{AB} := |\psi\rangle\langle\psi|$ pure state.

$$\Rightarrow S(A,B) = -\mathrm{tr}\{|\psi\rangle\langle\psi| \ln(|\psi\rangle\langle\psi|)\} = (\text{diagonalise})$$

$$= -\mathrm{tr}\{U D \log D\, U^+\} = -\log 1 = 0$$

since $\rho^{AB}$ is rank 1, hence $D = \begin{vmatrix} 1 & 0 & \cdots \\ & & 0 \end{vmatrix}$ $\square$

4) Let $\rho^{AB}$ be pure again. We can find a Schmidt decomp. for $|\psi\rangle$ such that:

$$\rho^{AB} = \sum_{ij} \sqrt{\lambda_i \lambda_j}\; |i^A\rangle\langle j^A| \otimes |i^B\rangle\langle j^B|$$

for $\{|i^A\rangle\}$, $\{|i^B\rangle\}$ ONB for $H_A, H_B$ resp.

$$\Rightarrow S(A) = S(\mathrm{tr}_B(\rho^{AB})) =$$

$$= S\Big(\sum_i \lambda_i\, |i^A\rangle\langle i^A|\Big)$$

$$S(B) = S\Big(\sum_i \lambda_i\, |i^B\rangle\langle i^B|\Big)$$

$\Rightarrow$ $\rho^A$ and $\rho^B$ have the same eigenvalues.

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$$\Rightarrow \quad S(A) = S(B) = -\sum_i \lambda_i \ln \lambda_i$$

$\square$

5) We can use again the inequality in the previous point 2, with the choices:

$$\rho := \rho^A \in B(H_A) \qquad \sigma := \frac{1\!\!1}{n}$$

with $n = \dim(H_A)$

$$\Rightarrow \quad 0 \leq -S(A) - \operatorname{tr}\left(\rho^A \ln \frac{1\!\!1}{n}\right) =$$

(consider the diagon. $\rho^A = UDU^\dagger$)

$$= -\operatorname{tr}\left(D \ln \frac{1\!\!1}{n}\right) = -\underbrace{\sum_{i=1}^{n} \lambda_i(\rho^A)}_{1} \cdot \ln \frac{1}{n} = + \ln n$$

$$\Rightarrow \quad S(A) \leq \ln n$$

Then one can easily see that the highest entropy corresponds to the completely depolarised state $\rho^A := \frac{1\!\!1}{n}$.

$\square$

## exercise 3

To prove the 2-step cooling inequality we can exploit Lemma 4.5 in the lecture notes.

Let $T_1, T_2, T_3$ be quantum channels and suppose we have two achievable rates:

$$c_1 > \lim_{J\to\infty} \frac{N_J}{M_J} \qquad \text{for } T_2 \text{ with resp. to } T_1$$

$$c_2 > \lim_{J\to\infty} \frac{M_J}{K_J} \qquad \text{" } T_3 \text{ " " " } T_2$$

as they are achievable rates, we also have that:

$$\lim_{J\to\infty} \Delta\left(T_2^{\otimes M_J}, T_1^{\otimes N_J}\right) = 0$$
$$\lim_{J\to\infty} \Delta\left(T_3^{\otimes K_J}, T_2^{\otimes M_J}\right) = 0$$

Among all the possible choices for $\{K_J\}_J$, we pick a ~~sequence~~ strictly increasing sequence such that $\lim_{J\to\infty} \frac{K_{J+1}}{K_J} = 1$.

Moreover, note that:

$$\| T_1^{\otimes N} - E_1 E_2 T_3^{\otimes K} D_2 D_1 \|_{cb} =$$

$$= \| T_1^{\otimes N} - E_1 T_2^{\otimes M} D_1 + E_1 T_2^{\otimes M} D_1 - E_1 E_2 T_3^{\otimes K} D_2 D_1 \|_{cb} \leq$$

$$\leq \| T_1^{\otimes N} - E_1 T_2^{\otimes M} D_1 \|_{cb} + \underbrace{\cancel{\| E_1 \|_{cb}}}_{1} \| T_2^{\otimes M} - E_2 T_3^{\otimes K} D_2 \|_{cb} \underbrace{\cancel{\| D_1 \|_{cb}}}_{1}$$

where we used that the $\|\cdot\|_{cb}$ of a quantum channel is one.

This result means that we also have:

$$\lim_{J\to\infty} \Delta\left(T_3^{\otimes K_J}, T_1^{\otimes N_J}\right) = 0$$

Therefore, by Lemma 4.6, we know that any

$c < \liminf_{J\to\infty} \frac{N_J}{K_J}$ is an achievable rate for $T_3$ with resp. to $T_1$. Hence, $\liminf_{J\to\infty} \frac{N_J}{K_J}$ gives a lower bound for $c\,(T_3, T_1)$.

But then we have:

$$\liminf_{J\to\infty} \frac{N_J}{K_J} \leq \lim_{J\to\infty} \frac{N_J}{M_J} \cdot \lim_{h\to\infty} \frac{M_h}{K_h} < c_1 c_2$$

$$\Rightarrow \quad c\,(T_3, T_1) \geq c\,(T_2, T_1)\, c\,(T_3, T_2)$$

$\square$

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exercise 4

To prove the "bottle neck" inequality, first note that

$$\| 1_B^{\otimes N} - \underbrace{E\,(T_2T_1)^{\otimes M} D}_{①} \|_{cb} = \| 1_B^{\otimes N} - (E T_2^{\otimes M})\, T_1^{\otimes M} D \|_{cb} \quad ②$$

$$= \| 1_B^{\otimes N} - E\, T_2^{\otimes M} (T_1^{\otimes M} D) \|_{cb} \quad ③$$

where $E T_2^{\otimes M}$ is a possible encoding for $T_1^{\otimes M}$, while $T_1^{\otimes M} D$ is a possible decoding for $T_2^{\otimes M}$.

Now, $\forall \{N_\gamma\}_\gamma, \{M_\gamma\}_\gamma$ s.t.

$\lim_{\gamma\to\infty} \Delta(①) = 0$ we have $\lim_{\gamma\to\infty} \Delta(②) = 0$

$\lim_{\gamma\to\infty} \Delta(③) = 0$

Therefore, achievable rates for $T_2T_1$ are also achievable rates for $T_1$ and $T_2$.
By the way, the supremum of the ach. rates for $T_2T_1$ is not equal to the supremum of ach. rates for $T_1$ and for $T_2$ separately, in general.

Therefore,

$$C\,(T_2T_1, 1_B) \le \min \{ C(T_1, 1_B), C(T_2, 1_B) \}$$

□