sheet 1

exercise 1

1) $A^\dagger A \geq 0$, then we can find a spectral decomposition
$A^\dagger A = UDU^\dagger$, where U unitary and D diagonal.
moreover, from positivity we know that $\mathrm{eig}_i(A^\dagger A) \geq 0$.
Then, we can define $J := \sqrt{A^\dagger A} = U\sqrt{D}\,U^\dagger$, where $\sqrt{D}$ is defined componentwise.
J is also positive, so it admits a spectral decomposition:

$$J = \sum_i \lambda_i |e_i\rangle\langle e_i| = \sum_i \lambda_i e_i \tilde{e}_i \quad , \quad \{e_i\} \text{ orthonormal basis}$$

where in the first sum we have the bra/ket notation, and in the second sum $\tilde{e}_i$ is the corresponding element of e_i in the dual space.
These two notations are equivalent.
Define $|\psi_i\rangle = A|e_i\rangle$; we want to find an orthonormal basis.
Note $\langle\psi_j|\psi_i\rangle = \langle e_j|A^\dagger A|e_i\rangle = \langle e_j|J^2|e_i\rangle = \lambda_i^2\delta_{ij}$

or, equivalently

$$\tilde{\psi}_j(\psi_i) = \tilde{e}_j J^2 e_i = \sum_k \lambda_k^2 \underbrace{\tilde{e}_j(e_k)}_{\delta_{jk}} \underbrace{\tilde{e}_k(e_i)}_{\delta_{ki}} = \lambda_i^2 \delta_{ij}$$

Consider $\lambda_i \neq 0$ only and define for them $|\phi_i\rangle = \frac{|\psi_i\rangle}{\lambda_i}$. Then, $\{|\phi_i\rangle\}$ forms an orthonormal set, that can be extended to an orthonormal basis (Gram-Schmidt).
Now, to prove $A = UJ$, ~~we need to~~ note first that $J|e_i\rangle = \lambda_i|e_i\rangle$
Then we seek a U such that:

$$\underset{\lambda_i U|e_i\rangle}{\underset{\parallel}{UJ|e_i\rangle}} = A|e_i\rangle = |\psi_i\rangle$$

and so it is evident that we can define: $U := \sum_i |\phi_i\rangle\langle e_i|$

$\cancel{U := \sum_i \frac{1}{\lambda_i}|\psi_i\rangle\langle e_i| = \sum_i |\phi_i\rangle\langle e_i| \quad \lambda_i \neq 0}$

exploiting the orthonormal basis built before.

U is obviously unitary: $U^{+}U = \sum_{ij} |e_i\rangle \underbrace{\langle\phi_i|\phi_j\rangle}_{\delta_{ij}} \langle e_j| =$

$= \sum_i |e_i\rangle\langle e_i| = \mathbb{1}$ (same for $UU^{+} = \mathbb{1}$)

uniqueness: J is always unique as $A^{+}A = JU^{+}UJ = J^2$ and from our definition of J: J^2 unique $\Rightarrow J$ unique

~~If~~ A is full rank $\Rightarrow A$ invertible $\Rightarrow J$ invertible

then $U = AJ^{-1}$ and so U unique.

$\square$

2) Let's use the polar decomposition.

$A = UJ$, U unitary , J positive

as $J \geq 0$ it admits spectral decomposition and so

$J = VDV^{+}$, V unitary , D diagonal

therefore, $A = UJ = \underbrace{UV}_{\tilde{U}} DV^{+} = \tilde{U} D V^{+}$

$\square$

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exercise 2

As $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$, we can express it as

$$|\psi\rangle = \sum_{ij} a_{ij} \, |e_i\rangle \otimes |e_j\rangle$$

with $\{|e_i\rangle\}$ ~~[illegible]~~ basis for $\mathbb{C}^d$.

Let $a_{ij} = [A]_{ij}$, where $A \in \mathbb{C}^{d\times d}$; then we can use the SVD of exercise 1.2 to get:

$$a_{ij} = u_{ik} \, d_{kk} \, v_{kj}$$

with u_{ik}, v_{kj} elements of unitary matrices and $d_{kk} \geq 0$, $\forall k = 1, \dots, d$.

Define $\lambda_k := d_{kk}$, then

$$|\psi\rangle = \sum_{ijk} u_{ik} \, d_{kk} \, v_{kj} \, |e_i\rangle \otimes |e_j\rangle =$$

$$= \sum_k \lambda_k \, \underbrace{\Big(\sum_i u_{ik} |e_i\rangle\Big)}_{\substack{\shortparallel \\ |k_A\rangle}} \otimes \underbrace{\Big(\sum_j v_{kj} |e_j\rangle\Big)}_{\substack{\shortparallel \\ |k_B\rangle}} = \sum_k \lambda_k \, |k_A\rangle \otimes |k_B\rangle$$

$|k_A\rangle$ and $|k_B\rangle$ form respectively two orthonormal sets, as

$$\langle k_A | k'_A \rangle = \sum_i u^*_{ik} \, u_{ik'} = [U^\dagger U]_{kk'} = \delta_{k,k'} \qquad (\text{same for } |k_B\rangle)$$

$\square$

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exercise 3

Prove the isomorphism $\mathrm{Hom}(\mathcal{H}, \mathcal{H}') \cong \mathcal{H}^* \otimes \mathcal{H}'$.

Recall that $\tilde{v} \in \mathcal{H}^* \Rightarrow \exists\, v \in \mathcal{H}$ s.t. $\tilde{v}(w) = \langle v, w\rangle$
$\forall w \in \mathcal{H}$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathcal{H}$.

Now, we will define a map φ between the two spaces, show that it is well-defined and linear, and finally show that there exists also φ^{-1}, well-defined and linear too. Then φ will be the isomorphism that proves the statement.

Therefore, define the map

$$\varphi : \mathcal{H}^* \otimes \mathcal{H}' \longrightarrow \mathrm{Hom}(\mathcal{H}, \mathcal{H}')$$
$$\tilde{v} \otimes w' \longmapsto \tilde{v}(\cdot)\, w' = \langle v, \cdot \rangle w'$$

for every element in $\{\tilde{v}_i\}, \{w_i'\}$ basis for $\mathcal{H}^*$ and $\mathcal{H}'$ respectively. Then you can extend by linearity to all the elements in the space.

Let $u, u' \in \mathcal{H}$, $\alpha, \beta \in \mathbb{C}$:

$$\varphi(\tilde{v} \otimes w')(\alpha u + \beta u') = \alpha\, \varphi(\tilde{v} \otimes w')(u) + \beta\, \varphi(\tilde{v} \otimes w')(u')$$

as this holds $\forall u, u' \in \mathcal{H}$, $\forall \alpha, \beta \in \mathbb{C} \rightarrow \varphi(\tilde{v} \otimes w') \in \mathrm{Hom}(\mathcal{H}, \mathcal{H}')$

and hence φ is well-defined.

By construction, φ is also linear.

Now, let $\psi : \mathcal{H} \to \mathcal{H}'$ be a linear map and denote with $\{e_i\}, \{e_i'\}$ basis for $\mathcal{H}$ and $\mathcal{H}'$ respectively. Then,

$$\psi(e_i) = \sum_j \underbrace{a_j(e_i)}_{\in \mathbb{C}} e_j'$$

It is easy to see that $a_\gamma(e_i)$ is a linear functional and hence, by the representation theorem,

$\exists!\, v_\gamma \in \mathfrak{h} \;:\; a_\gamma(e_i) = \langle v_\gamma, e_i \rangle$

This suggests the following definition:

$$\varphi^{-1}: \mathrm{Hom}(\mathfrak{h}, \mathfrak{h}') \longrightarrow \mathfrak{h}^* \otimes \mathfrak{h}'$$
$$\psi = \sum_\gamma a_\gamma(\cdot)\, e_\gamma' \longmapsto \sum_\gamma \tilde{v}_\gamma \otimes e_\gamma'$$

By construction, this is linear and well-defined.

Then, a simple calculation shows that:

$$\varphi \circ \varphi^{-1} = id_{\mathrm{Hom}(\mathfrak{h}, \mathfrak{h}')}$$
$$\varphi^{-1} \circ \varphi = id_{\mathfrak{h}^* \otimes \mathfrak{h}'}$$

hence φ^{-1} is the inverse of φ, ~~and~~ thus φ is an isomorphism.

$\square$