Foundations of Quantum Information Theory

Sheet 5

Discussion: 18.05.2015

Exercise 1: Consider a finite dimensional vector space E with a proper, generating cone E_+ containing a base $\Sigma \subset E_+$. Show that

$$\begin{aligned} \varphi: \quad & E^* \to \mathit{Aff}(\Sigma) \\ & f \mapsto f \restriction_\Sigma \end{aligned}$$

is an order isomorphism from E^* equipped with the dual cone E^*_+ onto $\mathit{Aff}(\Sigma)$ with the natural cone

$$\mathit{Aff}_+(\Sigma) = \{f \in \mathit{Aff}(\Sigma) \mid f(x) \geq 0, \ \forall x \in \Sigma\}$$

(6 pts)

Exercise 2: Consider a compact (non-empty) convex set $\mathcal{C} \subset E \cong \mathbb{R}^n$ with non-vanishing interior and $0 \in \mathcal{C}$; then, consider the set

$$\Sigma = \{\delta_x \in \mathit{Aff}^*(\mathcal{C}) \mid x \in \mathcal{C}, \ \delta_x(f) = f(x)\}$$

which is, according to 1.8 and 1.10 in the notes, convex and isomorphic (as a convex structure) to $\mathcal{C}$. Show that Σ coincides with the base of $\mathit{Aff}^*_+(\mathcal{C})$ given by

$$\Sigma_{\mathbb{I}} = \left\{\varphi \in \mathit{Aff}^*_+(\mathcal{C}) \mid \varphi(\mathbb{I}) = 1\right\}$$

where $\mathbb{I} \in \mathit{Aff}(\mathcal{C})$ such that $\mathbb{I}(x) = 1, \ \forall x \in \mathcal{C}$. *(7 pts)*

Exercise 3: (Minkowski Product) Consider the spaces:

(a) $\mathcal{B}_{sa}(\mathbb{C}^2) := \{A \in \mathcal{B}(\mathbb{C}^2) | A^* = A\}$ with the cone $\mathcal{B}_+(\mathbb{C}^2) := \{A \in \mathcal{B}(\mathbb{C}^2) | A \geq 0\}$

(b) $\mathbb{R}^4$ with the cone $C^+ := \{x \in \mathbb{R}^4 | x_0^2 - \sum_{j=1}^3 x_j^2 \geq 0, x_0 \geq 0\}$

1. Show that both spaces are isomorphic as ordered vector spaces, i.e. construct a linear isomorphism $T : \mathbb{R}^4 \to \mathcal{B}_{sa}(\mathbb{C}^2)$ such that $T(C^+) = \mathcal{B}_+(\mathbb{C}^2)$. *(4 pts)*

2. Show that a linear isomorphism $T : \mathbb{R}^4 \to \mathbb{R}^4$ is an order automorphism (i.e. $T(C^+) = C^+$ or anti-automorphism (i.e. $T(C^+) = -C^+$) if and only if

$$\eta(Tx, Ty) = c\eta(x, y)$$

where c is a positive constant and η denotes the Minkowski product,

$$\eta(x, y) = x_0 y_0 - \sum_{j=1}^{3} x_j y_j$$

(Hint: What happens to lightlike vectors, i.e. $x \in \mathbb{R}^4$ s.th. $\eta(x, x) = 0$?) - (6 pts)

Exercise 4: (Ordered Vector Spaces) Let $\mathcal{H}$ be a finite dimensional Hilbert space. Equip the space $\mathcal{B}_{sa}(\mathcal{H})$ of self-adjoint matrices with the operator norm $\|A\|_\infty := \sup_{\|v\|_2=1} \|Av\|_2$.

1. Show that $\mathcal{B}_{sa}(\mathcal{H})$ together with this norm, the natural ordering and the identity is an order unit space. *(3 pts)*

2. Compute the dual base norm space to $\mathcal{B}_{sa}(\mathcal{H})$ with its positive cone, its base and its base norm. *(4 pts)*