Foundations of Quantum Information Theory

Sheet 3

Discussion: 04.05.2015

Exercise 1: (States and effects)

1. Define the set of 2×2 density operators as $\mathcal{S}_{2\times 2} := \{\rho \in \mathcal{B}(\mathbb{C}^2) \mid \rho = \rho^\dagger; \rho \geq 0; \mathrm{tr}(\rho) = 1\}$; define then the Bloch sphere as $\mathcal{S}_{Bloch} := \{x \in \mathbb{R}^3 \mid \|x\| \leq 1\}$. Prove the isomorphism:

$$\mathcal{S}_{2\times 2} \cong \mathcal{S}_{Bloch}$$

   (3 pt)

2. Calculate the extremal points of the set of density matrices (set of states) in the case of arbitrary dimension. *(4 pt)*

3. Calculate the extremal points of the set of effects $\mathcal{A} := \{A \in \mathcal{B}(\mathbb{C}^n) \mid 0 \leq A \leq \mathbb{I}, n \in \mathbb{R}\}$, where $\mathbb{I}$ is the identity matrix. *(3 pt)*

Exercise 2: (Convex Sets and Inequalities) Prove that, if $\mathcal{C} \subset \mathbb{R}^n$ is a closed, bounded and convex set, then $\mathcal{C}$ is an arbitrary (i.e. not necessarily finite) intersection of halfspaces defined by inequalities. *(4 pt)*

Exercise 3: (Convex Functions) Let $f : \mathbb{R}^n \supset \mathcal{S} \to \mathbb{R} \cup \{\pm\infty\}$, define the *epigraph*

$$\{(x, \mu) | x \in \mathcal{S}, \mu \in \mathbb{R}, \mu \geq f(x)\}$$

We define f to be *convex* if the epigraph of f is a convex subset of $\mathbb{R}^{n+1}$.

1. Prove that if $\mathcal{S}$ is convex, then f is convex if and only if

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y), \qquad 0 \leq \lambda \leq 1$$

   for all $x, y \in \mathcal{S}$. *(3 pt)*

2. Let f be a convex function. Prove that for every open subset $\mathcal{C} \subset \mathcal{S}$, f is continuous on $\mathcal{C}$. *(5 pt)*

Exercise 4: (Minkowski Functional) Let $\mathcal{V}$ be a vector space over the reals. We call a subset $\mathcal{C} \subset \mathcal{V}$ *absorbing*, if for every $x \in \mathcal{V}$ there exists $t \geq 0$ such that $tx \in \mathcal{C}$. If

$\mathcal{C}$ is convex, it is called *balanced* if $x \in \mathcal{C}$ if and only if $-x \in \mathcal{C}$.
Given $\mathcal{C} \in \mathcal{V}$, define the *Minkowski functional*:

$$\begin{aligned} \rho : \mathcal{V} &\to [0, \infty) \\ x &\mapsto \inf\{\lambda | x \in \lambda C\} \end{aligned}$$

1. Prove that if $\mathcal{C}$ is a convex, absorbing, balanced set, then ρ is a seminorm (homogeneous + triangle inequality) *(4 pts)*.

2. Prove that for any p-norm on $\mathcal{V} = \mathbb{R}^n$ (i.e. $\|v\|_p := (\sum_i |v_i|^p)^{1/p}$), the unit ball $\mathcal{B} \subset \mathcal{V}$ is convex, absorbing and balanced, hence defines a seminorm via the Minkowski functional. Show furthermore that $\rho(\cdot) = \|\cdot\|_p$ in this case *(4 pts)*.