Information Theory [MA5103]
Homework 5

1. Consider binary channels with additive noise: $Y_i = X_i + Z_i$ where $(X_1, \dots, X_n)$ and $(Z_1, \dots, Z_n)$ are independent. However, Z_i are not necessarily independent of each other while they are identically distributed with $p_{Z_i}(0) = p$ and $p_{Z_i}(1) = 1 - p$. Prove that

$$\max_{P_{X_1 \dots X_n}} I(X_1 \dots X_n | Y_1 \dots Y_n) \geq n(1 - H(\{p, 1-p\}))$$

   Can we interpret this in terms of channel capacity?

2. We prove the source-channel coding theorem. Suppose $\{V_i\}_{i \in \mathbb{N}}$ is a stationary stochastic process which satisfies the asymptotic equipartition property with respect to the entropy rate $H = H(\{V_i\}_{i \in \mathbb{N}})$. Prove the following the two statements.

   (a) (achievability) If $H < C$ then there exists a source-channel code with probability of error $p(\hat{V}^{(n)} \neq V^{(n)}) \to 0$. Hint: we think of source coding and channel coding separately.

   (b) (converse) If $H > C$ there is $\delta > 0$ such that $p(\hat{V}^{(n)} \neq V^{(n)}) > \delta$ for all source-channel codes and $n \in \mathbb{N}$. Hint: -is in the proofs for the converse part of noisy channel coding theorem or the feedback capacity.

3. We prove that shared randomness does not increase the channel capacity. Suppose we have a discrete memoryless channel with input alphabet $\mathcal{X}$ and output alphabet $\mathcal{Y}$. An (M, n) code with shared randomness Z with its range $\mathcal{Z}$ is a pair of functions $f : \{1, \dots, M\} \times \mathcal{Z} \to \mathcal{X}^n$ and $g : \mathcal{Y}^n \times \mathcal{Z} \to \{1, \dots, M\}$. Here, Z is independent of the channel or the message to be transmitted. Then, R is called an achievable rate if for all $\epsilon > 0$ there exists a $(\lceil 2^{nR} \rceil, n)$ code with shared randomness such that the error probability $p_e^{(n)}$ converges to 0 as $n \to \infty$. Prove that $R \leq C$. Hint: -is in the proofs for the converse part of noisy channel coding theorem or the feedback capacity.

4. * Let X and X' be i.i.d. random variables with finite range. Show that

$$Prob(X = X') \geq 2^{-H(X)}$$

*This is an additional question which can be solved by some formula from the past lectures.