Information Theory [MA5103]
Homework 4

The capacity C of a channel $P_{Y|X}$ is defined by the maximal mutual information:

$$C := \max_{p(x)} I(X : Y)$$

Here, $p(x, y) = p(y|x)p(x)$.

1. Let $\mathcal{X} = \mathcal{Y} = \{0, 1\}$ and consider the channel given by the conditional probabilities $p_{Y|X}(0|0) = 1$, $p_{Y|X}(1|0) = 0$ and $p_{Y|X}(0|1) = p_{Y|X}(1|1) = \frac{1}{2}$. Determine the channel capacity as well as the probability distribution on the input alphabet $\mathcal{X}$ achieving the capacity.

2. Let $\mathcal{X} = \mathcal{Y} = \{0, 1, \ldots, k-1\}$ considered as a group with addition modulo k, and consider the following channel. For an input random variable X the output is $Y = X + N$ where N is an $\mathcal{X}$-valued random variable which is independent of X (additive noise). Prove that the capacity C is written as
$$C = \log k - H(N)$$
and determine the capacity of this channel under each of the following conditions.
   (a) N is identically 0; the channel is the identity map.
   (b) The channel is binary symmetric ($k = 2$) with a bit flip probability p.
   (c) N is uniformly distributed over $\{0, 1, \ldots, m-1\}$ where $m \leq k$ is fixed.

3. Let $|\mathcal{X}| < \infty$, $\mathcal{Y} = \mathcal{X} \sqcup \{*\}$ (disjoint union), and consider the erasure channel given by
$$p(y|x) = \begin{cases} p & \text{if } y = x \\ 1-p & \text{if } y = * \\ 0 & \text{otherwise} \end{cases}$$
Calculate the capacity of this channel.

4. Let $X \to Y \to Z$ form a Markov chain which is considered to be the composite of two channels. I.e., these two channels are described by $P_{Y|X}$ and $P_{Z|Y}$ so that X is the input, Y is the output of the first channel and Z is that of the second. Prove that the capacity of the composite channel is bounded from above by the capacities of both channels. Can we generalize this to more than two channels?

Note: We will cover the question 4 from Homework 3 as well.