Information Theory [MA5103]
Homework 1

1. Prove that for any triple X, Y, Z of discrete random variables the Shannon entropy $H(\cdot)$ satisfies the inequality
$$H(XYZ) + H(Z) \leq H(XZ) + H(YZ)$$
called the *strong subadditivity* property.
Note that this is equivalent to positivity of the conditional mutual information:
$$I(X : Y|Z) = H(XZ) + H(YZ) - H(XYZ) - H(Z).$$
Hint: Expand $I(X : Y|Z)$ and write it as the convex combination with $p(z)$.

2. For $\alpha \geq 0$, $\alpha \neq 1$ we define the *Rényi entropy of order* α to be
$$H_\alpha(X) \equiv H_\alpha(p_1, \ldots, p_n) = \frac{1}{1-\alpha} \log \left(\sum_{i=1}^{n} p_i^\alpha \right)$$
where $(p_i)_{i=1}^n$ are the probabilities of the possible values of X.
Show that
   a) $\forall \pi \in S_n : H_\alpha(p_1, \ldots, p_n) = H_\alpha(p_{\pi(1)}, \ldots, p_{\pi(n)})$
   b) $H_\alpha(p_1, \ldots, p_n) = H_\alpha(p_1, \ldots, p_n, 0)$
   c) $H_\alpha(X, Y) = H_\alpha(X) + H_\alpha(Y)$ if X and Y are independent
   d) If p_i take only the two values: 0 or some constant, then show that $H_\alpha(X)$ is constant for all α.

   Remark: The Rényi shares important properties with the Shannon entropy. In fact, as $\alpha \to 1$, it converges to the Shannon entropy.

3. Let X_0 and X_1 be random variables such that they don't have any common value, and let Θ be a $\{0, 1\}$-valued random variable which is independent of both X_0 and X_1. Then $Y = X_\Theta$ is also a random variable. Show that
$$H(X_\Theta) = pH(X_0) + (1-p)H(X_1) + H(\{p, 1-p\})$$

4. Let X_0 and X_1 be random variables with values in the same set, and let Θ be a $\{0, 1\}$-valued random variable which is independent of both X_0 and X_1. Then $Y = X_\Theta$ is also a random variable. Show that
$$H(Y) \geq pH(X_0) + (1-p)H(X_1)$$
where $p = Prob(\theta = 0)$, with equality iff $p \in \{0, 1\}$ or X_0 and X_1 are identically distributed.