10 Teleportation

We saw already that classical teleportation is impossible, if we accept no-signalling and the violation of the CHSH inequality. In the following we will see that a small but crucial modification namely *entanglement assisted teleportation* is possible.

Lemma 48. *For $d \in \mathbb{N}$ consider the set of d^2 unitaries*

$$U_j := \sum_{r=0}^{d-1} \eta^{r j_2} |r + j_1\rangle\langle r|, \;\; \eta := e^{\frac{2\pi i}{d}}, \;\; j = (j_1, j_2) \in \mathbb{Z}_d \times \mathbb{Z}_d$$

where addition inside the ket is modulo d and $\{|x\rangle \in \mathbb{C}^d\}_{x \in \mathbb{Z}_d}$ is an orthonormal basis. Then

(i) $\{U_j\}$ is a basis of operators in $\mathbb{C}^{d \times d}$ that is orthogonal w.r.t. the Hilbert-Schmidt inner product. More specifically, $\operatorname{tr}[U_i^* U_j] = d\,\delta_{i,j}$.

(ii) $U_i U_j = \eta^{i_2 j_1} U_{i+j}$ (with addition modulo d). Thus $U_j^{-1} = \eta^{j_1 j_2} U_{-j}$.

(iii) For $d = 2$ the set reduces to the set of Pauli matrices with identity, i.e., $(\mathbb{1}, \sigma_x, \sigma_y, \sigma_z) = (U_{(0,0)}, U_{(1,0)}, iU_{(1,1)}, U_{(0,1)})$.

Corollary 49. *Let $|\Omega\rangle := \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k\rangle \otimes |k\rangle$ be maximally entangled. Then with $j \in \mathbb{Z}_d \times \mathbb{Z}_d$ the set $|\Omega_j\rangle := (U_j \otimes \mathbb{1})|\Omega\rangle$ is an orthonormal basis of maximal entangled states in $\mathbb{C}^d \otimes \mathbb{C}^d$.*

Proof. The Ω_j's are orthonormal due to $\langle \Omega_i | \Omega_j \rangle = \operatorname{tr}[U_i^* U_j \otimes \mathbb{1} |\Omega\rangle\langle\Omega|] = \frac{1}{d} \operatorname{tr}[U_i^* U_j] = \delta_{i,j}$. Since there are d^2 Ω_j's, they form an orthonormal basis. □

Consider the following scenario described on $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_B$ with $\mathcal{H}_{A_1} \simeq \mathcal{H}_{A_2} \simeq \mathcal{H}_B$: Assume $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$ and $\mathcal{H}_B$ describe Alice's and Bob's part respectively and that the initial state is pure and described by $|\Psi\rangle_{A_1} \otimes |\Omega\rangle_{A_2 B}$, where $|\Omega\rangle := \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k\rangle \otimes |k\rangle$ is maximally entangled.
The task is now to "teleport" the state $|\Psi\rangle$ from Alice to Bob by

(i) performing a measurement on Alice's side

(ii) sending the measurement outcome to Bob

(iii) performing an operation on Bob's side.

None of these steps is allowed to depend on Ψ.

(i) The measurement:
Take $S = \mathbb{Z}_d \times \mathbb{Z}_d$ and $M(j) = |\Omega_j\rangle\langle\Omega_j|$ where $|\Omega_j\rangle := (U_j \otimes \mathbb{1})|\Omega\rangle$.
We want to determine the state ρ_j on Bob's side conditioned on Alice having obtained outcome j. Suppose this happens with probability p_j.
Then with $\overline{\underline{\Psi}} := |\Psi\rangle\langle\Psi|,\ \omega := |\Omega\rangle\langle\Omega|$

$$\begin{aligned}
\operatorname{tr}[\rho_j B] &= \frac{1}{p_j} \operatorname{tr}\left[\left(\overline{\underline{\Psi}}_{A_1} \otimes \omega_{A_2 B}\right)(M_j \otimes B)\right] \\
&= \frac{1}{p_j} \operatorname{tr}\left[\left(\left(U_j^* \overline{\underline{\Psi}} U_j\right)_{A_1} \otimes \omega_{A_2 B}\right)(\omega_{A_1 A_2} \otimes B)\right] \\
&= \frac{1}{d^2 p_j} \operatorname{tr}\left[U_j^* \overline{\underline{\Psi}} U_j B\right] \ \forall B \in \mathbb{C}^{d \times d}.
\end{aligned}$$

Applying this to $B = \mathbb{1}$, we see that $p_j = \frac{1}{d^2}$ independent of Φ.
Hence after the measurement Bob's state is $\rho_j = U_j^* \overline{\underline{\Psi}} U_j$ if the outcome was j.

(ii)-(iii) If Alice informs Bob about the measurement outcome, he can apply the channel $T_j(\rho) := U_j \rho U_j^*$ so that finally he possesses the state $\overline{\underline{\Psi}}$.