(1) Prove that a bipartite pure state is entangled iff it is not a product.

(2) Show that a convex combination of mutually orthogonal maximally entangled states can be separable.

(3) a) Define $|\Omega\rangle := \sum_{i=1}^{d} e_i \otimes f_i \in \mathbb{C}^d \otimes \mathbb{C}^d$ where $\{e_i\}, \{f_i\}$ are ONBs.
Prove that $W := \mathbb{1} - |\Omega\rangle\langle\Omega|$ is an entanglement witness.

b) Consider the one-parameter family of density operators
$$\rho_t := t \frac{\mathbb{1}}{d^2} + (1-t)\frac{|\Omega\rangle\langle\Omega|}{d} \in \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d), \quad t \in [0,1]$$
For which parameter range can you show that ρ_t is entangled?

(4) Define a linear map $\phi: \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$
$$\phi(X) := \mathrm{tr}[X]\,\mathbb{1} - X$$

a) Prove that ϕ is positive but not completely positive.

b) Construct an entanglement criterion using ϕ.

(5) Consider $\rho := |\psi\rangle\langle\psi|$ with $|\psi\rangle := \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right) \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$.

a) Regarding this as $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ where $\mathcal{H}_B = \mathbb{C}^2 \otimes \mathbb{C}^2$, is ρ entangled?

b) If we trace out one of the three qubits is the remaining reduced density operator $\mathrm{tr}_3[\rho] \in \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ entangled?