Throughout Hilbert spaces $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2$ are over $\mathbb{C}$ and all norms are the ones induced by the inner product.

1. Read the section *Hilbert spaces* in the lecture notes!
2. Show that the closed unit ball of any Hilbert space is strictly convex.
3. Show that any linear map $U : \mathcal{H}_1 \to \mathcal{H}_2$ between Hilbert spaces that preserves the norm also preserves the corresponding inner products.
4. a) Prove that $\psi = \varphi$ iff $\forall \phi \in \mathcal{H} : \langle \phi, \varphi \rangle = \langle \phi, \psi \rangle$.

   b) Let $A : \mathcal{H} \to \mathcal{H}$ be linear, $\psi, \varphi \in \mathcal{H}$. Verify the identity

$$\langle \varphi, A\psi \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \langle \psi + i^k \varphi, A(\psi + i^k \varphi) \rangle.$$

   c) Let $A, B : \mathcal{H} \to \mathcal{H}$ be linear. Show that $A = B$ iff $\forall \psi \in \mathcal{H} : \langle \psi, A\psi \rangle = \langle \psi, B\psi \rangle$.
5. Prove that every separable, infinite dimensional Hilbert space is isomorphic to $l_2(\mathbb{N})$.