Quantum Effects [MA5047]

Sheet 2

Discussion: 12.11.2013

Exercise 1: (Wagner's polynomial matrix identity)

1. Let $A, B, C \in \mathbb{C}^{2\times 2}$ be arbitrary. Prove that
$$[[A,B]^2, C] = 0$$
2. Is this true if $A, B, C \in \mathbb{C}^{3\times 3}$? Either present a proof or a counterexample.

Exercise 2: (Robertson's uncertainty relation for n observables)
Let $A_1, \dots, A_n \in \mathcal{B}(\mathcal{H})$ with some Hilbert space $\mathcal{H}$ be Hermitian and $\psi \in \mathcal{H}$. Define matrices

$$\sigma_{kl} := \frac{i}{2}\langle\psi|[A_k, A_l]|\psi\rangle, \quad V_{kl} := \frac{1}{2}\langle\psi|\{A_k - \langle A_k\rangle\mathbb{1}, A_l - \langle A_l\rangle\mathbb{1}\}_+|\psi\rangle$$

where $\{X, Y\}_+ := XY + YX$ is the *anti-commutator*. Show:

1. $V \geq i\sigma$
2. $\det(V) \geq \det(\sigma)$
3. for $n = 2$, 2. implies the uncertainty relation proven in the lecture for pure states.

Exercise 3:
Let $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be a density operator. Then

1. If $\{e_i \in \mathcal{H}_A\}$ and $\{f_l \in \mathcal{H}_B\}$ are orthonormal bases, then
$$\langle e_i, \rho_A e_j\rangle = \sum_l \langle e_i \otimes f_l, \rho\, e_j \otimes f_l\rangle$$
where ρ_A is the reduced density operator with respect to $\mathcal{H}_A$.
2. If $\rho = \rho_A \otimes \rho_B$, then ρ_A is the reduced density operator of ρ with respect to $\mathcal{H}_A$
3. If $\rho = |\phi\rangle\langle\phi|$ and $\phi = \sum_i \sqrt{\lambda_i} e_i \otimes f_i$ is a Schmidt decomposition, then $\rho_A = \sum_i \lambda_i |e_i\rangle\langle e_i|$ and $\rho_B = \sum_i \lambda_i |f_i\rangle\langle f_i|$ are the reduced density operators with respect to $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.