# Quantum Effects [MA5047]

Sheet 1

Discussion: 29.10.2013

**Exercise 1: (physical interpretation of the trace-norm distance)** Consider a separable Hilbert space $\mathcal{H}$ with the bounded operators $\mathcal{B}(\mathcal{H})$. Then

1. **Lemma:** Given an operator $P \in \mathcal{B}(\mathcal{H})$ and a state $\rho \in \mathcal{S}_1 \subset \mathcal{B}(\mathcal{H})$ with $0 \leq P \leq \mathbb{1}$, show that

$$0 \leq \mathrm{tr}[\rho P] \leq 1$$

2. **Lemma:** Given two density matrices $\rho_1, \rho_2 \in \mathcal{S}_1$, and define $D_\pm := (\rho_1 - \rho_2)_\pm$ the positive (+) and negative (-) part of $\rho_1 - \rho_2$, i.e. $D_\pm \geq 0$, $D_+ D_- = 0$ and $D_+ - D_- = \rho_1 - \rho_2$, then show that

$$\mathrm{tr}(D_\pm) = \frac{1}{2}\|\rho_1 - \rho_2\|_1$$

   where $\|\cdot\|_1$ denotes the trace-norm on $\mathcal{S}_1$.

3. Prove that the trace-norm quantifies the distinguishability of quantum states in the following sense: Assume there are two preparation devices that prepare $\rho_1, \rho_2 \in \mathcal{S}_1$ respectively. We are now given one of the two preparation devices with probability 1/2 and should decide which. For this task, we are allowed to use any two-outcome POVM we want, i.e. a POVM $M$ with the discrete outcome set $\mathcal{S} := \{1, 2\}$ and $M(1) + M(2) = \mathbb{1}$. If we assign the outcome 1 to the hypothesis that our preparation device prepares $\rho_1$ and 2 to the hypothesis that it prepares $\rho_2$, then the average probability (over many experiments) that we make a wrong prediction is given by:

$$p_e := \frac{1}{2}(\mathrm{tr}[\rho_1 M(2)] + \mathrm{tr}[\rho_2 M(1)])$$

   Prove that

$$p_e \geq \frac{1}{2}\left(1 - \frac{1}{2}\|\rho_1 - \rho_2\|_1\right)$$

   and that equality can be obtained for a special choice of the POVM.

You may assume $\dim(\mathcal{H}) < \infty$ during the exercise.

**Exercise 2:** Given a separable Hilbert space $\mathcal{H}$ and operators $A, B \in \mathcal{B}(\mathcal{H})$, define the *commutator* via

$$[A, B] := AB - BA$$

Then show the following relations:

1. $\mathrm{tr}[A, B] = 0$ if $AB, BA \in \mathcal{S}_1(\mathcal{H})$.
2. $\mathrm{tr}[\rho[A, B]] \leq \|[A, B]\| \leq 2\|A\|\|B\|$ for all states $\rho \in \mathcal{S}_1(\mathcal{H})$

**Exercise 3: (Bloch-sphere-picture of qubits)** In this exercise, we visualize states on $\mathcal{H} = \mathbb{C}^2$. First introduce the *Pauli-matrices*:

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and write $\vec{\sigma} := (\sigma_1, \sigma_2, \sigma_3)$.

1. Show that every state $\rho \in \mathcal{B}(\mathcal{H})$ has a unique decomposition

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{v} \cdot \vec{\sigma}) = \frac{1}{2}\left(\mathbb{1} + \sum_{i=1}^{3} v_i \sigma_i\right)$$

   with $\vec{v} \in \mathbb{R}^3$ and $\|\vec{v}\|^2 = \sum_{i=1}^3 v_i^2 \leq 1$. This implies that there is a one-to-one correspondence between states on $\mathbb{C}^2$ and points of the (closed) unit ball $\mathcal{B}_1 \subseteq \mathbb{R}^3$.

2. Show that all pure states lie on the *Bloch-sphere*, i.e. satisfy $\|\vec{v}\| = 1$.