Hausaufgaben

2.1. Embeddings in l_2

Give a counterexample (as simple as possible) of the following proposition: Any countable metric space admits an isometric embedding in l_2 (or even l_p, $p \in (1, \infty)$).

HINT: There are basic geometric properties of the l_2 unit ball that need not hold for arbitrary metric spaces.

2.2. Compatible distances

a) Let A be a real symmetric $n \times n$ matrix. Show that A is positive semi-definite iff there exists an $n \times n$ real matrix X such that $A = XX^T$.

b) Let m_{ij}, $i, j = 1, ..., n$, be positive real numbers such that $m_{ij} = m_{ji}$ for all i, j and $m_{ii} = 0$ for all i. Show that there exist points $p_0, ..., p_n \in \mathbb{R}^n$ such that $\|p_i - p_j\|_2 = m_{ij}$, $i, j = 1, ..., n$, iff the matrix $G_{ij} := \frac{1}{2}(m_{0i}^2 + m_{0j}^2 - m_{ij}^2)$ is positive semi-definite. HINT: Consider the identity $\langle x, y\rangle = \frac{1}{2}(\|x\|_2^2 + \|y\|_2^2 - \|x - y\|_2^2)$, $x, y \in \mathbb{R}^n$, for the points $x_i = p_i - p_0$.

2.3. Equilateral sets

Let $X \simeq \mathbb{R}^n$ be a Banach space. A set $M \subseteq X$ is called equilateral set iff the distance between any two points of the set is the same.

Prove that for $X = l_1^n$ there is no equilateral set $M \subseteq l_1^n$ with $|M| > 100n^4$. HINT:

a) Prove that for every natural numbers d, q there exists a map $f_{d,q} : [0, 1]^d \to \mathbb{R}^{dq}$ such that for every $x, y \in [0, 1]^d$

$$\|x - y\|_1 - \frac{2d}{q} \leq \frac{1}{q}\|f_{d,q}(x) - f_{d,q}(y)\|_2^2 \leq \|x - y\|_1 + \frac{2d}{q}.$$

b) Make a proof by contradiction assuming the following proposition is true:
Let $p_1, .., p_d \subseteq \mathbb{R}^n$ be points such that for every $i \neq j$ we have

$$1 - \frac{1}{\sqrt{d}} \leq \|p_i - p_j\|_2^2 \leq 1 + \frac{1}{\sqrt{d}}.$$

Then $d \leq 2(n + 2)$.