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Asymptotic Geometric Analysis
http://www-m5.ma.tum.de/Allgemeines/MA5024_2015W

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Blatt 2
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Hausaufgaben

2.1. Haar Measure

Let $L \in G_{n,k}$ be a k-dimensional subspace of $\mathbb{R}^n$ and let $U \in O(n)$ be a Haar-random rotation. Show that UL is a Haar-random k-dimensional subspace.

2.2. Concentration of Measure

a) Let $A := \{x \in S^{n-1} : x \leq 0\}$ be the southern hemisphere and $A_\delta := \{x \in S^{n-1} : \exists y \in A : \|x - y\| \leq \delta\}$ be the complement of a 'spherical cap' for $\delta \in [0, 1]$. Provide an elementary provable bound on the measure $1 - \sigma(A_\delta)$ of the spherical cap.

b) Argue that this implies concentration of measure around any set B with $\sigma(B) = 1/2$, if we assume the isoperimetric theorem for the sphere: Among all sets of fixed measure on S^{n-1}, the spherical cap has the smallest boundary measure.

c) Let $f : S^{n-1} \to \mathbb{R}$ be 1-Lipschitz and $m := \mathrm{med}(f)$. Prove an upper bound for $|\mathbb{E}(f^2) - m^2|$ in terms of n.

d) Let $f : \mathbb{R}^n \to \mathbb{R}$ be 1-Lipschitz and γ the Gaussian measure on $\mathbb{R}^n$. Derive a lower bound for $\gamma(|f - \mathrm{med}(f)| \leq \delta)$ in terms of δ.
HINT: Use the geometric Gaussian measure concentration result stated in the lecture.

2.3. Johnson-Lindenstrauss

a) What can be said about the preservation of angles in the Johnson-Lindenstrauss theorem?

b) Let $\{u_1, ..., u_n\} \subseteq \mathbb{R}^n$ be an orthonormal basis. Show that for any $\epsilon \in (0, 1)$ there is a linear map $L : \mathbb{R}^n \to \mathbb{R}^k$ with $k = O(\log(n)/\epsilon^2)$ s.t. $|\langle L(u_i), L(u_j)\rangle| \leq \epsilon\|L(u_i)\|\|L(u_j)\|$ for all $i \neq j$.