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Asymptotic Geometric Analysis
http://www-m5.ma.tum.de/Allgemeines/MA5024_2015W

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Blatt 2
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Hausaufgaben

2.1. Banach-Mazur distance

a) Show that the geometric and analytic version of the Banach-Mazur Distance coincide. That is, if $B_X,\ B_Y$ are unit balls of normed spaces $X \simeq Y \simeq \mathbb{R}^n$, then $d(B_X, B_Y) = d(X,Y)$.

b) Show that $X \simeq Y \simeq \mathbb{R}^n$ are isometrically isomorphic iff $d(X,Y) = 1$.

c) Let $1 \leq p \leq q \leq 2$ or $2 \leq p \leq q \leq \infty$. Show that for every $n \in \mathbb{N}$:

$$d(l_p^n, l_q^n) = n^{1/p-1/q}.$$

HINT: For " $\geq$ " use $d(l_2^n, l_\infty^n) = \sqrt{n}$ together with the submultiplicativity of the Banach-Mazur distance.

2.2. Brunn-Minkovski inequality

a) The Prékopa-Leidler inequality states that $\|h\|_1 \geq \|f\|_1^{1-\lambda}\|g\|_1^{\lambda}$ whenever $h, f, g : \mathbb{R}^n \to \mathbb{R}^+$ are Lebesgue integrable and for $\lambda \in (0,1)$ statisfy:

$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(x)^{\lambda}.$$

Prove the Brunn-Minkovski inequality from the Prékopa-Leidler inequality.

b) For a Lebesgue measurable set $K \in \mathbb{R}^n$ define the centroid of K by

$$x(K) := \frac{1}{vol(K)} \int_K y dy.$$

Let $C := \{x \in \mathbb{R}^n :\ x_1 \in [0,h], \sum_{i=1}^{n-1} x_i^2 \leq x_1^2\}$ be the standard cone of hight h. Let $\overline{x}$ be the centroid of C. Let $L := C \cap \{x \in \mathbb{R}^n : x_1 \leq \overline{x}_1\}$. Show that $\frac{1}{2} \leq \frac{vol(L)}{vol(C)} \leq \frac{1}{e}$.

c) Prove Grünbaum's Theorem: Let $K \subseteq \mathbb{R}^n$ be a convex body and divide K into $K_1,\ K_2$ using a hyperplane. If K_1 contains the centroid of K, then

$$\frac{vol(K_1)}{vol(K)} \geq 1/e.$$

HINT: Assume w.l.o.g. that the centroid lies in the origin and that the hyperplane is given by $\{x \in \mathbb{R}^n : x_1 = 0\}$.

- Construct a set K' as follows. Replace each slice $K_t = K \cap \{x \in \mathbb{R}^n : x_1 = t\}$ with a ball of the same volume. Show that K' is convex using the Brunn-Minkovski inequality.
- Show that, without increasing the ratio, K' can be deformed into a cone and use c).