An Introduction to Quantum Spin Systems[1]
Notes for MA5020 (John von Neumann Guest Lectures)

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Contents

10. Four Examples

The phenomena we studied in the general theorems and discussion in the previous chapters, spontaneaous symmetry breaking and long-range order, gapped versus gapless excitation spectrum, and the decay of correlations, are well-illustrated by the ground states of the isotropic ferromagnetic Heisenberg model (Example 1), the ferromagnetic XXZ model (Example 2), and the AKLT chain (Example 3). Later we will also discuss the Toric Code model to illustrate the concept of topological order (Example 4). All these models are examples of so-called Frustration-Free models, a property we will exploit in more detail in the next chapter on Matrix Product States and their Hamiltonians.

10.1. **Example 1: the isotropic Heisenberg model.** We start with the isotropic ferromagnetic spin 1/2 Heisenberg model on $\mathbb{Z}^\nu$, introduced by Heisenberg [30], which presents a good illustration of the gapless excitation spectrum implied by Goldstone's Theorem in the presence of spontaneous breaking of a continuous symmetry, in this case $SU(2)$.

At each $x \in \mathbb{Z}^2$, we have a spin 1/2 system with Hilbert space $\mathcal{H}_{\{x\}} \cong \mathbb{C}^2$. The interaction is between nearest neighbors only and is given in terms of the Pauli matrices by

$$h = -(\sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2 + \sigma^3 \otimes \sigma^3) = (\mathbb{1} - 2T) \in M_2 \otimes M_2, \tag{10.1}$$

where T denotes the transposition operator determined by $T(u \otimes v) = v \otimes u$, for all $u, v \in \mathbb{C}^2$. For every finite $\Lambda \subset \mathbb{Z}^\nu$, the Hamiltonian is given by

$$H_\Lambda = \sum_{\substack{x,y\in\Lambda \\ |x-y|=1}} (\mathbb{1} - 2T_{x,y}). \tag{10.2}$$

For example, one can take $\Lambda = [a_1, b_1] \times \cdots \times [a_\nu, b_\nu]$, with $a_i < b_i \in \mathbb{Z}$, $i = 1, \ldots, \nu$. It is quite obvious from the form Hamiltonian that all states symmetric under arbitrary permutations of the sites in Λ, will be a ground state of H_Λ. If Λ is connected (in the nearest neighbor sense), the set of transpositions $T_{x,y}$ for all nearest neighbor pairs $x, y \in \Lambda$, generate the full permutation group. In this case the ground state space of H_Λ is exactly the subspace of $\mathcal{H}_\Lambda$ consisting of all symmetric vectors. Let $E_0(H)$ denote the smalled eigenvalue of H. Then,

$$E_0(H_\Lambda) = -\#\{\{x, y\} \mid x, y \in \Lambda, |x - y|\}. \tag{10.3}$$

It will be slightly simpler to work with the non-negative definite Hamiltonian

$$0 \leq \tilde{H}_\Lambda = \frac{1}{2}(H_\Lambda - E_0(H_\Lambda)\mathbb{1}) = \sum_{\substack{x,y\in\Lambda \\ |x-y|=1}} (\mathbb{1} - T_{x,y}), \tag{10.4}$$

which obviously has the same ground state space and has H_Λ.

For all $u \in SU(2)$, we have $[T, u \otimes u] = 0$, and hence H_Λ (and $\tilde{H}_\Lambda$) commute with

$$U_\Lambda = \bigotimes_{x\in\Lambda} u. \tag{10.5}$$

As a consequence, the Hamiltonians also commute with the third component of the total magnetization or, equivalently, with the following operator N_Λ which counts the number of down spins (minus signs):

$$N_\Lambda = \sum_{x\in\Lambda} (\mathbb{1} - \sigma^3_x)/2. \tag{10.6}$$

spec$N_\Lambda = \{0, 1, \ldots, |\Lambda|\}$, and there is one symmetric state in each eigenspace of N_Λ. Hence, $\dim\ker(\tilde{H}_\Lambda) = |\Lambda| + 1$. For $n = 0, \ldots, |\Lambda|$, let $\mathcal{H}^{(n)}$ denote the eigenspace of N_Λ belonging to the eigenvalue n. This is the space of states with n down spins. A basis for this spaces can therefore be labeled by the possible sets of n sites (the positions of the down spins) in Λ. Hence,

$$\dim(\mathcal{H}_\Lambda^{(n)}) = \binom{|\Lambda|}{n}. \tag{10.7}$$

$\mathcal{H}_\Lambda^{(0)}$ is one-dimensional and spanned by the vector $\bigotimes_{x\in\Lambda} |+\rangle$. Let ω_+ denote the corresponding vector state. On each $\mathcal{A}_\Lambda$, ω_+ is uniquely determined by the property $\omega(\sigma_x^3) = 1$, for all $x \in \Lambda$. It follows that this also determines ω_+ uniquely on $\mathcal{A}_{\mathbb{Z}^\nu}$.

Since ω_+ is a product state its GNS representation is easy to construct. As GNS Hilbert space we can take $\ell^2(\mathcal{P}_0(\mathbb{Z}^\nu))$. Let $\{\xi_X \mid x \in \mathcal{P}_0(\mathbb{Z}^\nu)\}$ denote the standard orthonormal basis of $\ell^2(\mathcal{P}_0(\mathbb{Z}^\nu))$ given by Kronecker delta functions. The finite subsets $X \in \mathcal{P}_0(\mathbb{Z}^\nu)$ are the locations of a finite number of down spins. In ω_+, the are zero down spins and this state should therefore represented by $\Omega = \xi_\emptyset$. This can be verified with the following defintion of the representation π:

$$\pi_\omega(\sigma_x^-)\xi_X = \begin{cases} \xi_{X\cup\{x\}} & \text{if } x \notin X \\ 0 & \text{if } x \in X \end{cases}. \tag{10.8}$$

The morphism property of π implies that it is uniquely determined by its action on σ_x^-, $x \in \mathbb{Z}^\nu$, and one can easily derive its action on arbitrary local observables. For instance, $\pi(\sigma_x^+) = \pi(\sigma_x^-)^*$, $\pi(\sigma_x^3) = \pi(\sigma_x^+)\pi(\sigma_x^-) - \pi(\sigma_x^-)\pi(\sigma_x^+)$, etc.. Since we have $\pi(\prod_{x\in X}\sigma_x^-)\Omega = \xi_X$, it is clear that $\pi(\mathcal{A}_{\mathbb{Z}^\nu}^{\mathrm{loc}})\Omega$ is dense in $\mathcal{H}$. Therefore $(\mathcal{H}, \pi, \Omega)$ is the GNS triple for ω_+.

In the GNS representation of ω_+, the number of down spins of the infinite system is represented by the densely defined self-adjoint operator N, for which $\pi(\mathcal{A}_{\mathbb{Z}^\nu}^{\mathrm{loc}})\Omega$ is a core, and for which the standard basis vectors ξ_X are eigenvectors with eigenvalue $n = |X|$. The corresponding eigenspaces $\mathcal{H}^{(n)}$ are invariant subspaces of the GNS Hamiltonian H_{ω_+}. The latter is easily see from the explicit definition of H_{ω_+} on $\pi(\mathcal{A}_{\mathbb{Z}^\nu}^{\mathrm{loc}})\Omega$ as follows. For each $X \in \mathbb{Z}$, define $\overline{X} = \{y \in \mathbb{Z}^\nu \mid d(y, X) \leq 1\}$. Then, for all finite X, $\overline{X}$ is finite, and for all $A \in \mathcal{A}_X$, we have

$$H_{\omega_+}\pi(A)\Omega = \lim_{\Lambda\to\mathbb{Z}^\nu}[\pi(H_\Lambda), \pi(A)]\Omega = [\pi(H_{\overline{X}}), \pi(A)]\Omega = \sum_{\substack{x,y\in\overline{X} \\ |x-y|=1}} 2(\mathbb{1} - T_{x,y})\pi(A)\Omega, \tag{10.9}$$

where, by slight abuse of notation, $T_{x,y} = \pi(T_{x,y})$ is the transposition of the states at x and y acting as a unitary operator on $\mathcal{H}$, and we have used $T_{x,y}\Omega = \Omega$, for all $x, y \in \mathbb{Z}^\nu$. Clearly, the spaces $\mathcal{H}^{(n)}$ are invariant under the $T_{x,y}$, and therefore also invariant subspaces of H_{ω_+}. So, we have $\mathrm{spec}(H_{\omega_+} \restriction_{\mathcal{H}^{(1)}}) \subset \mathrm{spec}(H_{\omega_+})$. Now, $\mathcal{H}^{(1)} = \overline{\mathrm{span}\{\xi_{\{x\}} \mid x \in \mathbb{Z}^\nu\}} \cong \ell^2(\mathbb{Z}^\nu)$, and it turns out that $H_{\omega_+} \restriction_{\mathcal{H}^{(1)}}$ can be represented as a familiar operator on $\ell^2(\mathbb{Z}^\nu)$. To see this, calculate the matrix elements in the orthonormal basis $\{\xi_{\{x\}}\}$:

$$\langle \xi_{\{x\}}, H_{\omega_+}\xi_{\{y\}}\rangle = \begin{cases} -2 & \text{if } |x-y| = 1 \\ 4\nu & \text{if } x = y \\ 0 & \text{else} \end{cases}. \tag{10.10}$$

Up to trivial constants, these are the matrix elements of the discrete Laplacian on $\mathbb{Z}^\nu$. Its spectrum is the interval $[0, 8\nu]$, and is absolutely continuous. Therefore, H_{ω_+} has no gap above the ground state, as implied by the Goldstone Theorem. The generalized eigenfunction corresponding this part of the spectrum are called *spin waves*. Considering the positions of the single down spin in the subspace $\mathcal{H}^{(1)}$ as the coordinate of a particle, the corresponding generalized eigenfunctions are plane waves on $\mathbb{Z}^\nu$. Dyson pointed out that the generalized eigenstates of the Heisenberg ferromagnet in the subspace $\mathcal{H}^{(n)}$ can be regarded as describing n such particles with boson statistics which satisfy a hard-core condition (at most one particle can occupy any give site) and a nearest neighbor interaction. He also showed that the spin-S Heisenberg ferromagnet in the ground state representation can be described in a similar way as bosons and that the interaction would become weaker with increasing S [14,15]. Recently, Correggi, Giuliani, and Seiringer gave a rigorous proof that the approximation by free bosons is at least sufficiently good to predict correct low-temperature asymptotics of the free energy density of the Heisenberg model for any S [12].

10.2. **Example 2: the XXZ model.** The spin 1/2 ferromagnetic XXZ model depends on a real parameter, Δ, which is often called the anisotropy. It is defined in terms of a nearest neighbor interaction and coincides with the isotropic Heisenberg model for $\Delta = 1$. The XXZ nearest neighbor interaction is given by

$$h^{(\Delta)} = -(\sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2 + \Delta\sigma^3 \otimes \sigma^3) = h^{(1)} - (\Delta - 1)\sigma^3 \otimes \sigma^3. \tag{10.11}$$

We already studied the ground states of the isotropic model with interaction $h^{(1)}$ in the previous section. It is quite immediate that, for all $\Delta > 1$, there are exactly two of the ground states of the isotropic model that are simultaneous eigenstates belonging to the smallest each of the additional terms of the form $-(\Delta - 1)\sigma^3 \otimes \sigma^3$, namely the states with all spins up and all spins down. Let ω_+ and ω_- denote the corresponding states on $\mathcal{A}_{\mathbb{Z}^\nu}$. The XXZ model has a continuous symmetry, given by the rotations about the third axis, described by the group $U(1)$ (or $SO(2)$). This symmetry is, however not broken in the ground states of the model. The model has a discrete symmetry that is spontaneously broken: a $\mathbb{Z}_2$ symmetry represented by the automorphism α defined by

$$\alpha(A) = (\bigotimes_{x\in X} \sigma^1)A(\bigotimes_{x\in X} \sigma^1), \text{ for all } A \in \mathcal{A}_X, X \in \mathcal{P}_0(\mathbb{Z}^\nu). \tag{10.12}$$

One can show that the GNS Hamiltonian of $\omega_\pm$ have a spectral gap above their ground state equal to $2(\Delta - 1)$. As the Goldstone Theorem predicts, the continuous $U(1)$ is unbroken.

10.3. **Example 3: the AKLT model.** The AKLT model was introduced by Affleck, Kennedy, Lieb, and Tasaki [3,4]. It is a spin-1 chain, so $\mathcal{H}_x = \mathbb{C}^3$, for all $x \in \mathbb{Z}$, with an $SU(2)$-invariant nearest neighbor interaction. Its Hamiltonian for a finite chain of length $L \geq 2$ is given by

$$H_{[1,L]} = \sum_{x=1}^{L-1} \left(\frac{1}{3}\mathbb{1} + \frac{1}{2}\mathbf{S}_x \cdot \mathbf{S}_{x+1} + \frac{1}{6}(\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2\right) = \sum_{x=1}^{L-1} P^{(2)}_{x,x+1}. \tag{10.13}$$

Here, $\mathbf{S} = (S^1, S^2, S^3)$ is the vector of standard spin-1 matrices, and $P^{(2)}_{x,x+1} \in \mathcal{A}_{\{x,x+1\}}$ is the orthogonal projection onto the 5-dimensional subspace of $\mathbb{C}^3 \otimes \mathbb{C}^3$ corresponding to the spin-2 irreducible representation of $SU(2)$ contained in the tensor product of the two spin-1 representations acting on the sites x and $x + 1$.

In the limit of the infinite chain, the AKLT chain has a unique frustration-free ground state and there is a non-vanishing gap in the spectrum above the ground state. The correlation length, which is guaranteed to be finite by the Exponential Clustering Theorem, can be compute explicitly: $\xi = 1/\log 3$. Affleck, Kennedy, Lieb, and Tasaki proved these properties are proved in [4] model and thus demonstrated the existence of the so-called Haldane phase in quantum spin chains, predicted by Haldane based on his analysis of the large spin limit [27].

The exact ground state of the AKLT model has a special structure in which the correlations are generated by entangled nearest neighbor pairs and were called Valence Bond Solid states (VBS) in [3]. The construction of the ground state of the AKLT chain shows a close similarity with the Quantum Markov Chains constructed by Accardi [1]. Inspired by this similarity Fannes, Nachtergaele, and Werner introduced Finitely Correlated States (FCS) [17] and proved that FCS provide the exact ground state of a large family of spin chains, including the AKLT model, with similar properties. Matrix Product States (MPS) are a special case of FCS, and have proved to be a very useful tool in the study of quantum spin chains. Because of the matrix product structure of the formulas for the AKLT ground state given in [19], the name Matrix Product States was proposed in [35].

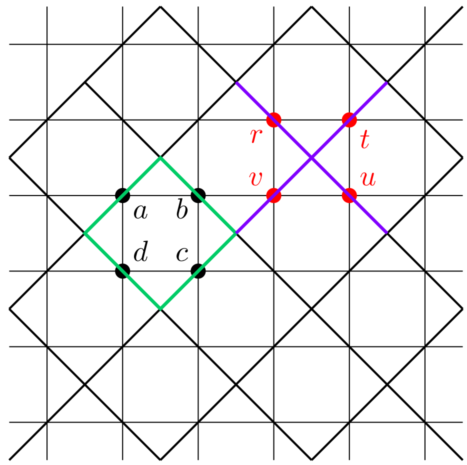

FIGURE 1. The distinction between stars and plaquettes becomes obvious by considering the spins to reside on the edges than rather than the vertices of a square lattice.

The ground state of the AKLT chain has only short-range correlations and no long-range order. Nevertheless, this state has a long-range structure that can be regarded as the one-dimensional analogue of Topological Order (see Example 4). This structure was discovered by den Nijs and Rommelse, who called it String Order [13].

There are several interesting generalizations of the AKLT model and its MPS ground state, including to higher dimensions, some of which we will discuss later on.

10.4. **Example 4: the Toric Code model.** The following example of a two-dimensional model which exhibits Topological Order was introduced by Kitaev [34]. One can associate the variables with the sites of the standard square lattice, but it turns out to be more convenient for the discussion of this model to associate the variables with the edges of a (different) square lattice. See Figure 10.4. The Hamiltonian contains two different types of four-body terms:

$$H_\Lambda = \sum_{s\subset\Lambda}(\mathbb{1} - A_s) + \sum_{p\subset\Lambda}(\mathbb{1} - B_p), \tag{10.14}$$

where s stands for 'star', meaning the four edges, labeled r, t, u, v, meeting a vertex, and p stands for 'plaquette', meaning four edges, labeled a, b, c, d, forming an square (see Figure). The corresponding terms in the Hamiltonian are defined as follows:

$$\begin{aligned} A_s &= \sigma^1_r\sigma^1_t\sigma^1_u\sigma^1_v \\ B_p &= \sigma^3_a\sigma^3_b\sigma^3_c\sigma^3_d. \end{aligned}$$

This model has the remarkable property that when defined on a finite lattice embedded in a closed two-dimensional manifold (e.g., a 2-torus) the ground state space has dimension 4^g, where g is the genus of the manifold ($g = 1$ for the 2-torus).

11. Frustration free models

In the past decade, much of the progress in our understanding of the ground state problem of quantum spin models was achieved by studying so-called frustration free interactions. An interaction $\Phi : \mathcal{P}_0(\Gamma) \to \mathcal{A}_{\text{loc}}$, is *frustration free* if for all $\Lambda \in \mathcal{P}_0(\Gamma)$ we have

$$\inf \operatorname{spec} \left(\sum_{X \subset \Lambda} \Phi(X) \right) = \sum_{X \subset \Lambda} \inf \operatorname{spec}(\Phi(X)). \tag{11.1}$$

Equivalently, the frustration free property can be expressed by stating that the finite-volume Hamiltonians $H_\Lambda = \sum_{X \subset \Lambda} \Phi(X)$ and each of the terms $\Phi(X)$ appearing in it, have a common eigenvector belonging to their respective smallest eigenvalues. While this requirement is generically *not* fulfilled, there is a remarkable range of quantum spin models, which model interesting physics, that do satisfy it. The frustration free property has turned out to be very helpful in the study of a range of problems. General results about the existence of a non-vanishing spectral gap in the thermodynamic limit and proving that a spectral gap above the ground state is stable under arbitrary perturbations of the interaction Φ, have so far only been proved for frustration free models. Understanding these questions first in the frustration free context, we expect, will prove to be an important step toward obtaining more general results.

Although the ferromagnetic Heisenberg model, introduced by Heisenberg in 1928 [30], is frustration-free, and other frustration-free models have been introduced a long time ago (e.g., the Majumdar-Ghosh model in 1969 [39–41]), frustration-free models as a class have been considered only more recently. The starting point for a wave of new developments was the introduction of the AKLT model by Affleck, Kennedy, Lieb, and Tasaki [3, 4], which is frustration free. The AKLT model played a pivotal role in more than one way. Not only was it a breakthrough by itself in establishing rigorously the existence of the so-called Haldane phase of quantum spin chains, it allso set in motion a series of fruitful new directions in the mathematics and physics research on quantum lattice models.

The AKLT model is a translation-invariant spin-1 chain with an $SU(2)$-invariant nearest neighbor interaction. For a finite chain of L spins the Hamiltonian of the AKLT model is given by

$$H_{[1,L]} = \sum_{x=1}^{L} \left(\frac{1}{3}\mathbb{1} + \frac{1}{2}\mathbf{S}_x \cdot \mathbf{S}_{x+1} + \frac{1}{6}(\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2 \right), \tag{11.2}$$

where $\mathbf{S}_x$ denotes the vector of the 3×3 spin-1 matrices acting at the xth site of the chain, which generate the three-dimensional unitary irreducible representation of $SU(2)$. See Appendix 9 for the definitions and elementary properties of representations of $SU(2)$. In [4] the authors prove that, the limit of the infinite chain, the ground state is unique, has a finite correlation length, and that there is a non-vanishing gap in the spectrum above the ground state, thereby demonstrating the existence of the Haldane phase of the isotropic spin-1 chain predicted by Haldane [26, 27]. This was a milestone result but the impact if this work and the AKLT model in particular, turned out to reach much further. Soon, it was realized that the exact ground state of the AKLT chain could be viewed as an example of a suitable generalization of the Quantum Markov Chains introduced by Accardi [1, 2]. This led to the introduction of Finitely Correlated States (FCS) [17, 19, 20], and the definition of a large class of frustration-free quantum spin models in one and more dimensions. Matrix Product States (MPS) is the name given to the subclass of Finitely Correlated States most relevant for the ground state problem in one dimension. An alternative representation goes under the name Valence Bond States (VBS), which is the term used in [4] for the particular $SU(2)$ invariant states constructed in that work. Examples of higher-dimensional VBS states already in the same paper and a general construction was given in [17]. Later, the construction was applied with considerable success to the study of concrete problems in two dimensions and renamed Products of Entangled Pairs (PEPS) in [55] which are, in turn, a special case of Tensor Networks [47].

Around the same time with the development of Matrix Product Sates, Steven White introduced his Density Matrix Renormalization Group method for the numerical computation of the ground state and low-lying excitations of quantum spin chains [**?**, 59]. The method immediately yielded very accurate results, e.g., for the spectral gap of the spin-1 antiferromagnetic Heisenberg chain and the AKLT chain [58]. The latter is easily understood since the exact ground state of the AKLT chain is a fixed point of the DMRG iteration [48]. By now we also understand why the DMRG method works well for one-dimensional problems more generally, especially for models with a non-vanishing gap, as is in the case for any Hamiltonian in the Haldane phase [5, 16, 28, 48].

The AKLT chain is frequently used as a testing ground for new concepts in many-body physics and quantum information theory. Well-known examples are string order [13], localizable entanglement [49], and symmetry-protected topological order [10, 23, 52, 54]. The model has been generalized in different directions: from spin 1 to higher spin [6], from one to two and more dimensions, from $SU(2)$ and $O(3)$ to $SU(n)$ and $O(2n+1)$ [53], etc. As already mentioned, the construction of the exact ground state of the AKLT chain led to the study of a large class of frustration-free models, which in turn provided a fruitful starting point for understanding gapped ground states, entanglement in many-body states (including the so-called Area Law), the development of numerical algorithms for calculating correlations, the spectrum of excitations, the dynamics, form factors etc., and the study of the complexity of the computational problem of finding the ground state of a quantum spin system and of estimating the spectral gap above it.

In the next section, we present the ground state of the AKLT chain in some detail. This will serve as a solid foundation for the general discussion of frustration free modes in the rest of this chapter.

11.1. **The AKLT chain.** The most general translation invariant spin-1 chain with an $SU(2)$-invariant nearest neighbor interaction is of the following form

$$H_{[1,L]} = \sum_{x=1}^{L} \left(J_0 \mathbb{1} + J_1 \mathbf{S}_x \cdot \mathbf{S}_{x+1} + J_2 (\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2 \right). \tag{11.3}$$

By a shift and a scaling of the energy, which has no consequences for the ground states, the three coupling constants can be taken to be $J_0 = 0, J_1 = \cos\theta, J_2 = \sin\theta$, with $\theta \in [0, 2\pi)$. The conjectures ground state phase diagram is depicted in Figure 11.1. The angle θ corresponding to the AKLT chain is given by $\tan\theta = 1/3$.

It is simple exercise using the irreducible representations of $SU(2)$ to show that the AKLT interaction is the orthogonal projections onto the spin-2 subspace of a pair of spins:

$$\frac{1}{3}\mathbb{1} + \frac{1}{2}\mathbf{S}_x \cdot \mathbf{S}_{x+1} + \frac{1}{6}(\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2 = P^{(2)}_{x,x+1}.$$

It follows that $\ker H_{[1,2]} = D^{(0)} \oplus D^{(1)} \subset \mathbb{C}^3 \otimes \mathbb{C}^3$. Therefore the space of ground state of the AKLT chain of length 2 is 4-dimensional and is given by the spin 0 and spin 1 vectors in the tensor product of two spins. It will turn out that $\ker H_{[1,2]}$ is 4-dimensional for all $L \geq 2$. In particular the ground state energy vanishes for all finite chains and the model is frustration free. To construct the ground states of the AKLT chain, one has the choice between the VBS, FCS, and MPS constructions mentioned above. As we will demonstrate below, these are three equivalent approaches that each highlight particular features of theses states.

From the decomposition $D^{(1)} \otimes D^{(1/2)} \cong D^{(1/2)} \oplus D^{(3/2)}$, it follows that there is an isometry $V : \mathbb{C}^2 \to \mathbb{C}^3 \otimes \mathbb{C}^2$, unique up to a phase, such that

$$V D^{(1/2)}(g) = (D^{(1)}(g) \otimes D^{(1/2)}(g))V, \text{ for all } g \in SU(2). \tag{11.4}$$

One says that V *intertwines* the $SU(2)$ representations $D^{(1/2)}$ and $D^{(1)} \otimes D^{(1/2)}$. With respect to the standard orthonormal basis of $\mathbb{C}^2$ and $\mathbb{C}^3 \otimes \mathbb{C}^2$ consisting of eigenvectors of the third component

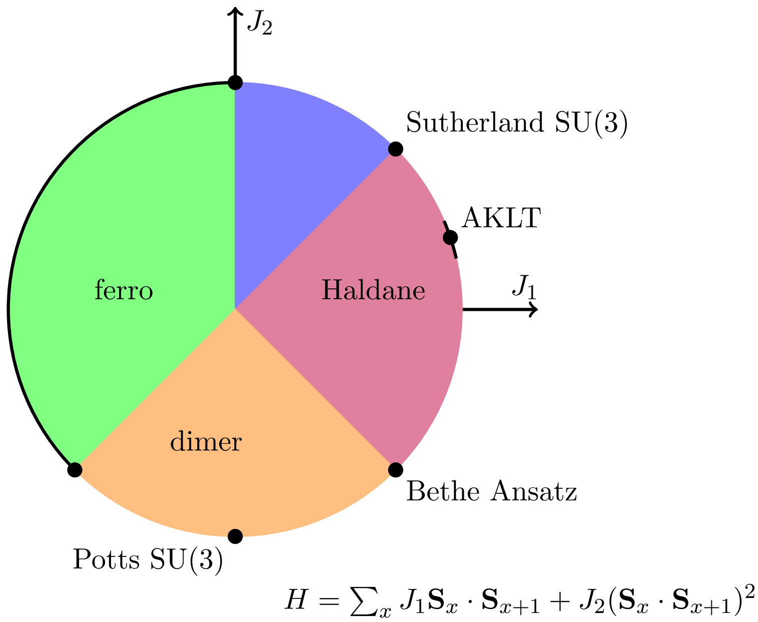

FIGURE 2. The conjectured phase diagram of the translation invariant spin 1 chains with $SU(2)$-invariant nearest neighbor interactions.

of the spin: $|1/2; m\rangle$, and $|1, 1/2; m_1, m_2\rangle$, the matrix elements of V are given by the familiar Clebsch-Gordan coefficients:

$$V|m\rangle = \sum_{\substack{m_1, m_2 \\ m_1 + m_2 = m}} \langle 1, 1/2; m_1, m_2 \mid 1/2; m\rangle |1, 1/2; m_1, m_2\rangle.$$

With the standard normalizations, $V^* V = \mathbb{1}$, i.e., V is an isometry.

For $\alpha, \beta \in \mathbb{C}^2$, and $n \geq 2$, define $\psi^{(n)}_{\alpha\beta} \in \mathcal{H}_{[1,n]}$ by

$$\psi^{(n)}_{\alpha\beta} = (\mathbb{1}_3)^{\otimes n} \otimes \langle \beta \mid (\underbrace{\mathbb{1}_3 \otimes \cdots \mathbb{1}_3}_{n-1} \otimes V) \cdots (\mathbb{1}_3 \otimes V) V |\alpha\rangle. \tag{11.5}$$

By using the interwining property of V n times, we find

$$\begin{aligned}
&(D^{(1)})^{\otimes n} \psi^{(n)}_{\alpha\beta} \\
&= (\mathbb{1}_3)^{\otimes n} \otimes \langle D^{(1/2)} \beta \mid (\underbrace{D^{(1)} \otimes \cdots D^{(1)}}_{n} \otimes D^{(1/2)} V) \cdots V |\alpha\rangle \\
&= (\mathbb{1}_3)^{\otimes n} \otimes \langle D^{(1/2)} \beta \mid (\underbrace{\mathbb{1}_3 \otimes \cdots \mathbb{1}_3}_{n-1} \otimes V) \cdots (\mathbb{1}_3 \otimes V) V | D^{(1/2)} \alpha\rangle.
\end{aligned} \tag{11.6}$$

This means that $SU(2)$ acts on the space $\{\psi^{(n)}_{\alpha\beta} \mid \alpha, \beta \in \mathbb{C}^2\}$ by the representation $(D^{(1/2)})^* \otimes D^{(1/2)} \cong D^{(0)} \oplus D^{(1)}$. In particular this proves that

$$P^{(2)} \psi^{(2)}_{\alpha\beta} = 0, \tag{11.7}$$

for all $\alpha, \beta \in \mathbb{C}^2$. Next, we will derive the MPS representation of the states $\psi^{(n)}_{\alpha\beta}$ and use it to show, in Proposition 11.1 below, that

$$\operatorname{span}\{\psi^{(n)}_{\alpha\beta} \mid \alpha, \beta \in \mathbb{C}^2\} = \ker H_{[1,n]}, \tag{11.8}$$

thus determining the ground state space of finite chains.

Given the standard basis $|1\rangle, |0\rangle, |-1\rangle$ of $\mathbb{C}^3$, we can define 2×2 matrices v_i, $i = 1, 0, -1$, by

$$V|\alpha\rangle = \sum_i |i\rangle \otimes v_i|\alpha\rangle.$$

Explicitly:

$$V = \begin{bmatrix} v_1 \\ v_0 \\ v_{-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \sqrt{\frac{2}{3}} & 0 \\ -\sqrt{\frac{1}{3}} & 0 \\ 0 & \sqrt{\frac{1}{3}} \\ 0 & -\sqrt{\frac{2}{3}} \\ 0 & 0 \end{bmatrix}. \tag{11.9}$$

One easily checks:

$$\langle i_1, \ldots, i_n \mid \psi^{(n)}_{\alpha\beta}\rangle = \langle \beta \mid v_{i_n} \cdots v_{i_1}|\alpha\rangle = \mathrm{Tr}|\alpha\rangle\langle\beta|\, v_{i_n} \cdots v_{i_1}. \tag{11.10}$$

In other words

$$\psi^{(n)}_{\alpha\beta} = \sum_{i_1,\ldots,i_n} \mathrm{Tr}[|\alpha\rangle\langle\beta| v_{i_n} \cdots v_{i_1}]|i_1, \ldots, i_n\rangle. \tag{11.11}$$

It therefore makes sense to extend the vectors $\psi^{(n)}$ linearly to a map $\psi^{(n)} : M_2 \to \mathcal{H}_{[1,n]}$:

$$\psi^{(n)}(B) = \sum_{i_1,\ldots,i_n} \mathrm{Tr}[Bv_{i_n} \cdots v_{i_1}]|i_1, \ldots, i_n\rangle,\ B \in M_2. \tag{11.12}$$

The following is then obvious:

$$\mathrm{span}\{\psi^{(n)}_{\alpha\beta} \mid \alpha, \beta \in \mathbb{C}^2\} = \{\psi^{(n)}(B) \mid B \in M_2\},$$

and we define $\mathcal{G}_n = \{\psi^{(n)}(B) \mid B \in M_2\}$. We can now prove the inclusion

$$\mathcal{G}_n \subset \ker H_{[1,n]}, \tag{11.13}$$

by a direct computation. Let $x = 1, \ldots, n-1$, and compute the expectation of $P^{(2)}_{x,x+1} \in \mathcal{A}_{[1,n]}$ in a state $\psi^{(n)}(B)$:

$$\begin{aligned}
&\langle \psi^{(n)}(B), P^{(2)}_{x,x+1}\psi^{(n)}(B)\rangle \\
&\quad = \sum_{\substack{i_1,\ldots,j_n \\ j_1,\ldots,j_n}} \overline{\mathrm{Tr}[Bv_{i_n} \cdots v_{i_1}]}\mathrm{Tr}[Bv_{j_n} \cdots v_{j_1}] \\
&\qquad\quad \times \langle i_1, \ldots, i_n|\mathbb{1}_{[1,x-1]} \otimes P^{(2)} \otimes \mathbb{1}_{x+2,n]}|j_1, \ldots, j_n\rangle \\
&\quad = \sum_{\substack{i_1,\ldots,i_{x-1} \\ i_{x+2},\ldots,i_n}} \langle \psi^{(2)}(v_{i_{x-1}} \cdots v_{i_1} B v_{i_n} \cdots v_{i_{x+2}}), P^{(2)}\psi^{(2)}(v_{i_{x-1}} \cdots v_{i_1} B v_{i_n} \cdots v_{i_{x+2}})\rangle
\end{aligned}$$

By linear extension we see from (11.7) that $P^{(2)}\psi^{(2)}(C) = 0$, for all $C \in M_2$. Hence, each term in the sum above vanishes. This proves (11.13). The other inclusion is proved as part of the following proposition.

Proposition 11.1. *For all $n \geq 2$ we have*

$$\ker H_{[1,n]} = \{\psi^{(n)}(B) \mid B \in M_2\}$$

Proof. We start by showing that

$$\dim\{\psi^{(2)}(B) \mid B \in M_2\} = 4.$$

To see this, recall the action of $SU(2)$ on the MPS vectors (11.7):

$$(D^{(1)} \otimes D^{(1)})\psi^{(2)}(B) = \psi^{(2)}((D^{(1/2)})^* B D^{(1/2)}).$$

Therefore, unless these vectors turn out to vanish, $\psi^{(2)}(\mathbb{1})$ is a singlet and $\psi^{(2)}(\sigma^i)$, $i = 1,2,3$, is a triplet representation of $SU(2)$. These are mutually orthogonal and the claim can be proved by showing they are non-zero. This follows by a straightforward computation using the definitions. E.g.,

$$\begin{aligned} \psi^{(2)}(\mathbb{1}) &= \mathrm{Tr}(v_{-1}v_1)|1,-1\rangle + \mathrm{Tr}(v_0)^2|0,0\rangle + \mathrm{Tr}(v_1 v_{-1})|-1,1\rangle \\ &= -\frac{2}{3}(|1,-1\rangle - |0,0\rangle + |-1,1\rangle. \end{aligned}$$

Therefore the singlet vector is non-zero. A similar computation shows that the triplet vectors are non-vanishing too. This also shows that the map $B \mapsto \psi^{(2)}(B)$ is injective. Furthermore, we can now show, by induction on n, that the maps $\psi^{(n)} : M_2 \to \mathcal{H}_{[1,n]}$ are injective for all $n \geq 2$. To do this, note that if $\psi^{(n)}$ is in injective, there exists a constant $c_n > 0$ such that $\|\psi^{(n)}(B)\|^2 \geq c_n \mathrm{Tr} B^* B$. Using the definition (11.12) we can then estimate $\|\psi^{(n+1)}(B)\|$ as follows:

$$\begin{aligned} \|\psi^{(n+1)}(B)\|^2 &= \sum_{i_{n+1}} \sum_{i_1,\ldots,i_n} |\mathrm{Tr} B v_{i_{n+1}} v_{i_n} \cdots v_{i_1}|^2 \\ &= \sum_{i_{n+1}} \|\psi^{(n)}(B v_{i_{n+1}})\|^2 \\ &\geq c_n \sum_{i_{n+1}} \mathrm{Tr}(B v_{i_{n+1}})^* B v_{i_{n+1}} \\ &= c_n \mathrm{Tr}\left[\sum_{i_{n+1}} v_{i_{n+1}} v_{i_{n+1}}^*\right] B^* B. \end{aligned}$$

Using the explicit form of the matrices v_i, (11.9), one easily checks that the sum in the square brackets equals $\mathbb{1}$. Since $c_n > 0$ by the induction hypothesis, this proves that $\psi^{(n+1)}$ is injective.

As the next step in the proof we show that

$$\ker H_{[1,3]} = (\mathcal{G}_2 \otimes \mathbb{C}^3) \cap (\mathbb{C}^3 \otimes \mathcal{G}_2) = \mathcal{G}_3. \tag{11.14}$$

The first identity follows directly from what we have proved so far and we already proved the inclusion of $\mathcal{G}_3 \subset \ker H_{[1,3]}$ in (11.13). We have shown above that $\psi^{(3)}$ is injective and, hence, $\dim \mathcal{G}_3 = 4$. Therefore, we will have proved our claim if we show that $\dim\ker H_{[1,3]} \leq 4$. We do this by considering the decomposition of $\mathcal{G}_2 \otimes \mathbb{C}^3$ into irreducible representations of $SU(2)$, and fact that for any $\phi \in \ker H_{[1,3]}$, we have $\phi \in \mathcal{G}_2 \otimes \mathbb{C}^3$ and $\mathbb{1} \otimes P^{(2)}\phi = 0$. $\mathcal{G}_2 = W_0 \oplus W_1$, where W_0 is a singlet and W_1 is a triplet for $SU(2)$. Therefore, in the decompostion of $\mathcal{G}_2 \otimes \mathbb{C}^3$ into irreducibles, we will have two triplets, one singlet, and one copy of the spin 2 representation. Using the methods explained in Appendix 9, one easily finds that the highest weight vector in that spin 2 representation is given by $\xi_{22} = |1,0,1\rangle - |0,1,1\rangle$. One easily checks that $(\mathbb{1} \otimes P^{(2)})\xi_{22} \neq 0$. Since there is only one spin 2 representation in the subspace $\mathcal{G}_2 \otimes \mathbb{C}^3$, it must therefore be orthogonal to $\ker H_{[1,3]}$. Similalry, one shows that the spin 1 highest weight vector $\xi_{11} = |1,-1,1\rangle - |0,0,1\rangle + |-1,1,1\rangle$ satsfies $(\mathbb{1} \otimes P^{(2)})\xi_{11} \neq 0$. Therefore, since there are only two copies of the spin 1 representation on $\mathcal{G}_2 \otimes \mathbb{C}^3$, at most one can belong to $\ker H_{[1,3]}$. Taking this together, we see that $\ker H_{[1,3]}$ can contain at most one singlet and one triplet representation and no representations of higher spin. It follows that $\dim\ker H_{[1,3]} \leq 4$, thus completing the proof of (11.14).

For the final step in the proof of the proposition, we will show the following intersection property of the spaces $\mathcal{G}_n$: for $\ell, r \geq 0, m \geq 2$,

$$(\mathcal{G}_{\ell+m} \otimes \mathcal{H}_{[1,r]}) \cap (\mathcal{H}_{[1,\ell]} \otimes \mathcal{G}_{m+r}) = \mathcal{G}_{\ell+m+r}. \tag{11.15}$$

It will be convenient to use multi-indices $\mathbf{i} = (i_1, \ldots, i_\ell)$, $\mathbf{j} = (j_1, \ldots, j_m)$, $\mathbf{k} = (k_1, \ldots, j_r)$, and to let $v_\mathbf{i}$ denote the product $v_{i_\ell} \cdots v_{i_1}$ etc. Then, for $\phi \in (\mathcal{G}_{\ell+m} \otimes \mathcal{H}_{[1,r]}) \cap (\mathcal{H}_{[1,\ell]} \otimes \mathcal{G}_{m+r})$, we have $C_\mathbf{i}, D_\mathbf{k} \in M_2$, such that

$$\phi = \sum_{\mathbf{i}} |\mathbf{i}\rangle \otimes \psi^{(m+r)}(C_\mathbf{i}) = \sum_{\mathbf{k}} \psi^{(\ell+m)}(D_\mathbf{k}) \otimes |\mathbf{k}\rangle. \tag{11.16}$$

By expanding both expressions using (11.12) and equating coefficients, one finds

$$0 = \mathrm{Tr} C_\mathbf{i} v_\mathbf{j} v_\mathbf{k} - \mathrm{Tr} D_\mathbf{k} v_\mathbf{i} v_\mathbf{j} = \mathrm{Tr}\,[v_\mathbf{k} C_\mathbf{i} - D_\mathbf{k} v_\mathbf{i}]\, v_\mathbf{j}.$$

This means that for all $\mathbf{i}, \mathbf{k}$ $\psi^{(m)}(v_\mathbf{k} C_\mathbf{i} - D_\mathbf{k} v_\mathbf{i}) = 0$. By the assumption $m \geq 2$, hence $\psi^{(m)}$ is injective and hence

$$v_\mathbf{k} C_\mathbf{i} - D_\mathbf{k} v_\mathbf{i} = 0, \text{ for all } \mathbf{i}, \mathbf{k}. \tag{11.17}$$

Multiplying this relation from the left by $v_\mathbf{k}^*$ and summing over $\mathbf{k}$, we find

$$\sum_{\mathbf{k}} (v_\mathbf{k}^* v_\mathbf{k}) C_\mathbf{i} = (\sum_{\mathbf{k}} v_\mathbf{k}^* D_\mathbf{k}) v_\mathbf{i}.$$

By the isometry property of V, we have $\sum_\mathbf{k} (v_\mathbf{k}^* v_\mathbf{k}) = \mathbb{1}$, and therefore

$$C_\mathbf{i} = B v_\mathbf{i}, \text{ with } B = \sum_{\mathbf{k}} v_\mathbf{k}^* D_\mathbf{k}.$$

Inserting this expression for $C_\mathbf{i}$ into (11.16) gives

$$\phi = \psi^{(\ell+m+r)}(B),$$

which completes the proof of (11.15).

We can now finish the proof of the proposition by combining the properties proved above:

$$\begin{aligned}
\ker H_{[1,n]} &= \bigcap_{x=1}^{n-1} \mathcal{H}_{[1,x-1]} \otimes \mathcal{G}_2 \otimes \mathcal{H}_{[x+2,n]} \\
&= \bigcap_{x=1}^{n-2} \mathcal{H}_{[1,x-1]} \otimes (\mathcal{G}_2 \otimes \mathbb{C}^3 \cap \mathbb{C}^3 \otimes \mathcal{G}_2) \otimes \mathcal{H}_{[x+3,n]} \\
&= \bigcap_{x=1}^{n-2} \mathcal{H}_{[1,x-1]} \otimes \mathcal{G}_3 \otimes \mathcal{H}_{[x+3,n]} \\
&= \bigcap_{x=1}^{n-3} \mathcal{H}_{[1,x-1]} \otimes \mathcal{G}_3 \otimes \mathcal{H}_{[x+4,n]} \cap \mathcal{G}_4 \\
&\vdots \\
&= \mathcal{G}_n
\end{aligned}$$

where we have used (11.14), (11.15), and the conventions that $\mathcal{H}_\emptyset = \mathbb{C}$ and $[a, b] = \emptyset$ if $a > b$. $\square$

The intersection property (11.15) is 'visualized' in the Valence Bond Solid representation of the ground states of the AKLT chain. This representation can be derived from the expression (11.5) by noting that, up to a normalization constant C, the intertwining isometry V can be expressed as follows: for all $u \in \mathbb{C}^2$

$$Vu = (P^+ \otimes \mathbb{1})(u \otimes \phi),$$

where $\phi \in \mathbb{C}^2 \otimes \mathbb{C}^2$ is the antisymmetric vector (i.e., the singlet state) and $P^+ : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^3$ is the projection onto the symmetric states, which represent the triplet of spin-1 states.

$$\begin{aligned}\psi^{(n)}_{\alpha\beta} &= \Big[\underbrace{P^+ \otimes \cdots \otimes P^+}_{n} \otimes \langle\beta|\Big]\Big[|\alpha\rangle \otimes \underbrace{\phi \otimes \phi \cdots \otimes \phi}_{n}\Big] \\ &= \pm\Big[\underbrace{P^+ \otimes \cdots \otimes P^+}_{n}\Big]\Big[|\alpha\rangle \otimes \underbrace{\phi \otimes \phi \cdots \otimes \phi}_{n-1} \otimes |-\beta\rangle\Big] \\ &= VBS\end{aligned}$$

Remark by bxn: VBS should be replaced by a visual representation of the VBS state as it routinely appears in the literature *End of Remark.*

To study correlation functions and the thermodynamic limit, the representation of the ground states as Finitely Correlated States is very convenient.

Since

$$\psi^{(n)}_{\alpha\beta} = \sum_{i_1,\ldots,i_n} \langle\beta \mid v_{i_n} \cdots v_{i_1}|\alpha\rangle |i_1,\ldots,i_n\rangle,$$

we have for $A_1,\ldots,A_n \in M_3$:

$$\begin{aligned}&\langle\psi^{(n)}_{\alpha\beta}|A_1 \otimes \cdots \otimes A_n \psi^{(n)}_{\alpha\beta}\rangle \\ =& \sum_{\substack{i_1,\ldots,i_n\\ j_1,\ldots,j_n}} (A_1)_{i_1j_1}\cdots(A_n)_{i_nj_n}\overline{\langle\beta \mid v_{i_n}\cdots v_{i_1}|\alpha\rangle}\langle\beta \mid v_{j_n}\cdots v_{j_1}|\alpha\rangle \\ =& \sum_{\substack{i_1,\ldots,i_n\\ j_1,\ldots,j_n}} (A_1)_{i_1j_1}\cdots(A_n)_{i_nj_n}\langle\alpha \mid v^*_{i_1}\cdots v^*_{i_n}|\beta\rangle\langle\beta \mid v_{j_n}\cdots v_{j_1}|\alpha\rangle \\ =& \sum_{\substack{i_1,\ldots,i_{n-1}\\ j_1,\ldots,j_{n-1}}} (A_1)_{i_1j_1}\cdots(A_n)_{i_{n-1}j_{n-1}} \\ & \times\langle\alpha \mid v^*_{i_1}\cdots v^*_{i_{n-1}}\Big[\sum_{i_n,j_n}(A_n)_{i_nj_n}v^*_{i_n}|\beta\rangle\langle\beta|v_{j_n}\Big]v_{j_{n-1}}\cdots v_{j_1}|\alpha\rangle.\end{aligned}$$

It is now convenient to define for all $A \in M_3$, a map $\mathbb{E}_A : M_2 \to M_2$ by

$$\mathbb{E}_A(B) = \sum_{ij} A_{ij} v^*_i B v_j = V^*(A \otimes B)V.$$

In terms of these maps we can write the expectations of general tensor product observables in a compact form:

$$\begin{aligned}&\langle\psi^{(n)}_{\alpha\beta}|A_1 \otimes \cdots \otimes A_n \psi^{(n)}_{\alpha\beta}\rangle \\ =& \sum_{\substack{i_1,\ldots,i_{n-1}\\ j_1,\ldots,j_{n-1}}} (A_1)_{i_1j_1}\cdots(A_n)_{i_{n-1}j_{n-1}}\langle\alpha \mid v^*_{i_1}\cdots v^*_{i_{n-1}}[\mathbb{E}_{A_n}(|\beta\rangle\langle\beta|)]v_{j_{n-1}}\cdots v_{j_1}|\alpha\rangle \\ =& \langle\alpha \mid \mathbb{E}_{A_1}\circ\cdots\circ\mathbb{E}_{A_n}(|\beta\rangle\langle\beta|)|\alpha\rangle.\end{aligned}$$

This expression makes the calculation of the thermodynamic limit very transparent. Adding $\ell+1$ sites to the left and r to the right of the interval $[1,n]$ gives the following expression for the expectation of $A = \mathbb{1}^{\otimes \ell} \otimes A_1 \otimes \cdots \otimes A_n \otimes \mathbb{1}^{\otimes r} \in \mathcal{A}_{[-\ell+1,n+r]}$ in the vector state $\psi_{\alpha\beta}^{(\ell+n+r)}$:

$$\begin{aligned}
&\langle \psi_{\alpha\beta}^{(\ell+n+r)} | A \psi_{\alpha\beta}^{(\ell+n+r)} \rangle \\
&= \langle \alpha \mid \mathbb{E}_{\mathbb{1}}^{\ell} \circ \mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n} \circ \mathbb{E}_{\mathbb{1}}^{r}(|\beta\rangle\langle\beta|)|\alpha\rangle \\
&= \mathrm{Tr}|\alpha\rangle\langle\alpha|\mathbb{E}_{\mathbb{1}}^{\ell} \circ \mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n} \circ \mathbb{E}_{\mathbb{1}}^{r}(|\beta\rangle\langle\beta|) \\
&= \mathrm{Tr}(\mathbb{E}_{\mathbb{1}}^{T})^{\ell}(|\alpha\rangle\langle\alpha|)\mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n} \circ \mathbb{E}_{\mathbb{1}}^{r}(|\beta\rangle\langle\beta|)
\end{aligned}$$

Here, $(\mathbb{E}_{\mathbb{1}}^{T})$ denotes the transpose of $\mathbb{E}_{\mathbb{1}}^{T}$ with respect to the Hilbert-Schmidt inner product on M_2.

The map $\mathbb{E}_{\mathbb{1}}$ is called the *transfer operator* and its spectral properties control the limits $\lim_{\ell\to\infty}$ and $\lim_{r\to\infty}$. Using (11.9) or (11.4) it is straightforward to verify the following diagonalization of $\mathbb{E}_{\mathbb{1}}$:

$$\mathbb{E}(\mathbb{1}) = \mathbb{1}, \quad \mathbb{E}_{\mathbb{1}}(\sigma^i) = -\frac{1}{3}\sigma^i, \; i = 1, 2, 3.$$

Since, for $B \in M_2$

$$B = \frac{1}{2}(\mathrm{Tr}B)\mathbb{1} + \frac{1}{2}\sum_{i=1}^{3}(\mathrm{Tr}B\sigma^i)\sigma^i,$$

we have

$$\mathbb{E}_{\mathbb{1}}(B) = \frac{1}{2}(\mathrm{Tr}B)\mathbb{1} - \frac{1}{3}[B - \frac{1}{2}(\mathrm{Tr}B)\mathbb{1}],$$

and therefore

$$\mathbb{E}_{\mathbb{1}}^{p}(|\beta\rangle\langle\beta|) = \frac{1}{2}\|\beta\|^2\mathbb{1} + \left(-\frac{1}{3}\right)^{p}[|\beta\rangle\langle\beta| - \frac{1}{2}\|\beta\|^2\mathbb{1}]. \tag{11.18}$$

This implies

$$\begin{aligned}
&\lim_{\ell\to\infty, r\to\infty} \langle \alpha \mid \mathbb{E}_{\mathbb{1}}^{\ell} \circ \mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n} \circ \mathbb{E}_{\mathbb{1}}^{r}(|\beta\rangle\langle\beta|)|\alpha\rangle \\
&= \frac{\|\alpha\|^2\|\beta\|^2}{2}\mathrm{Tr}(\frac{1}{2}\mathbb{1})\mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n}(\mathbb{1}) \\
&= \omega(A_1 \otimes \cdots \otimes A_n),
\end{aligned} \tag{11.19}$$

where ω is a translation invariant pure state on $\mathcal{A}_{\mathbb{Z}}$ uniquely determined by the above expression for simple tensor observables.

Define $Q : M_2 \to M_2$ by

$$Q(B) = \frac{1}{2}(\mathrm{Tr}B)\mathbb{1}.$$

(11.18) then implies

$$\|\mathbb{E}_{\mathbb{1}}^{p} - Q\| \le \frac{2}{3^p}.$$

By taking $A_2 = \cdots A_{n-1} = \mathbb{1}$ in (11.19), we obtain the following estimate for the two-point correlation function of ω:

$$\begin{aligned}
|\omega(A_1 \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes A_n) - \omega(A_1)\omega(A_n)| &= \left|\frac{1}{2}\mathrm{Tr}\mathbb{E}_{A_1} \circ (\mathbb{E}_{\mathbb{1}}^{n-2} - Q) \circ \mathbb{E}_{A_n}\right| \\
&\le \|A_1\|\|A_n\|\frac{C}{3^n}.
\end{aligned}$$

Thus we have shown exponential decay of correlations in the state ω. Next, we show that ω is the unique zero-energy ground state of the AKLT chain.

Proposition 11.2. *ω defined by (11.19) is the unique state on $\mathcal{A}_{\mathbb{Z}}$ such that $\omega(P_{x,x+1}^{(2)}) = 0$, for all $x \in \mathbb{Z}$.*

Proof. Any state η on $\mathcal{A}_{\mathbb{Z}}$ is uniquely determined by its restrictions to the subalgebras $\mathcal{A}_{[a,b]}$, $a < b$. Let $\rho_{[a,b]}$ denote the density matrices of η restricted to $\mathcal{A}_{[a,b]}$. $\eta(P^{(2)}_{x,x+1}) = 0$, for $x = a, \ldots, b-1$, implies that $\mathrm{ran}\rho_{[a,b]} \subset \mathcal{G}_{b-a+1}$. From (11.19) and Proposition 11.1 it then follows that, for all $a_1 < b_1 \in \mathbb{Z}$, $A_{a_1}, \ldots, A_{b_1} \in M_3$, we have

$$\begin{aligned}
&\eta(A_{a_1} \otimes \cdots \otimes A_{b_1}) \\
&\quad = \lim_{\substack{a \to -\infty \\ b \to \infty}} \mathrm{Tr}\rho_{[a,b]} \mathbb{1}_{[a,a_1-1]} \otimes A_{a_1} \otimes \cdots \otimes A_{b_1} \otimes \mathbb{1}_{[b_1+1,b]} \\
&\quad = \omega(A_{a_1} \otimes \cdots \otimes A_{b_1}),
\end{aligned}$$

proving the claim. □

So far, we have proved that the AKLT model has two of the three characterizing properties of the Haldane phase: it has a unique ground state and a finite correlation length. The third property, the non-vanishing spectral gap above the ground state, can be proved by further exploiting the structure of the ground states of the model. In order to avoid too much repetition, we will do this in the more general context of arbitrary quantum spin chains with a finite number of ground states that are of the MPS form (11.11).

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